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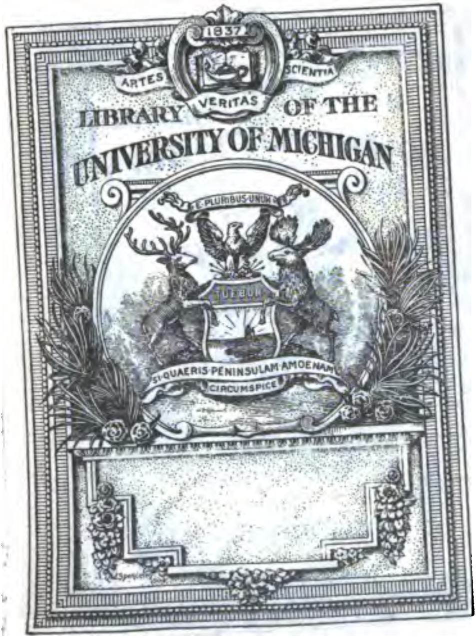

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ELEMENTS OF THE MATHEMATICAL
THEORY OF FLUID MOTION.

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WAVE AND VORTEX MOTION.

BY

THOMAS CRAIG, PH. D.,

Fellow in Physics in the Johns Hopkins University,
Baltimore, Md.

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PREFACE.

THE subject of Hydrodynamics embraces many of the most difficult problems in the range of physical research.

Although, at all times attracting the attention of the greatest minds, it is only within little over a century past that much real progress has been made in the solution of the many and complicated cases presented by the ordinary phenomena of Fluid Motion. The names of Euler, Lagrange and Laplace in the last century, and of Helmholtz, Stokes, Thomson, Rayleigh and Kirchhoff in this, stand out preeminently as those that have done the most to advance the theory to its present position. The object of the following article is to present in a short space the more important points in the Mathematical Theory of Fluid Motion, as it has been developed by these investigators. It is a want severely felt by any one making a study of this subject, that there exists no separate and complete treatise on Hydrodynamics.

It is a fact, I think, greatly to be regretted, that the men who do the most for the real advancement of science so seldom present to the world the result of their labors and extensive knowledge, in any other form than an occasional memoir in a scientific journal, or in a communication to a learned society. There are, however, notable exceptions to this general rule, as witness: Maxwell's treatise on Electricity and Magnetism; Rayleigh on Sound, Cayley's Elliptic Functions, and a few others. If some one would present to the public a treatise on Hydrodynamics, of the scope of those mentioned on other subjects, he would certainly receive the gratitude of all physical students, and confer a great boon upon the scientific world.

T. C.

Baltimore, *April*, 1879.

Elements of the Mathematical Theory

OF

FLUID MOTION.

THE following paper contains the mathematical investigation of some of the cases of the motion of incompressible, frictionless fluids. The results obtained are to be considered as applying only to this class of fluids, unless the contrary is expressly stated. The paper is intended to be introductory to a treatise which I hope before long to be able to publish.

In a subject so difficult as Hydrodynamics, there is but little chance for the discovery of hitherto unheard of properties of the quantities dealt with, so, in what follows, the reader will not look for much that is absolutely new in the way of fact, although the arrangement of the work and in many cases the methods employed are my own.

The references to the original sources from which information has been drawn, are given in every case, and I trust that these references, together with the matter contained in this paper, will prove of value to any one interested in the most difficult but beautiful problem of fluid motion.

Of late years, much has appeared in different places upon the subject of Hydrodynamics, but, so far as I am aware, there is no general work either in the English, French or German languages. The aim of this paper and the treatise which will follow will be to combine in one work, all of importance that has been written upon the subject, and so enable the student to forego the immense amount of research necessary in order thoroughly to inform himself upon any one branch of the subject.

The short section which appears upon the theory of the Potential, is principally taken from Clausius's work upon that subject. The references to theoretical mechanics are, unless otherwise stated, to Thomson and Tait's Natural Philosophy. Kirchhoff's Mathematische Physik, and Clifford's Elements of Dynamic, have also been consulted.

§ I.

GENERAL EQUATIONS OF FLUID MOTION.

Let X, Y, Z denote as usual the component forces acting at the point (x, y, z) of the fluid reckoned per unit of its mass —then denoting by ρ the density of the fluid we have for the forces acting upon the elementary mass $\rho dxdydz$ the expressions

$$X\rho dxdydz, \quad Y\rho dxdydz, \quad Z\rho dxdydz;$$

Now for the fluid pressure acting upon one face of the elementary parallelopiped, say, $\delta y\delta z$ we have $p\ \delta y\delta z$, p denoting the pressure on unit of area; upon the opposite face it is, neglecting powers of δx higher than the first,

$$-\delta y\delta z\left(p+\frac{dp}{dx}\delta x\right)$$

Consequently the resultant force due to fluid pressure acting in the direction of the axis x is,

$$-\delta y \delta z \frac{dp}{dx} \delta x.$$

The equilibrium of this portion of the fluid therefore requires that

$$\delta x \delta y \delta z \frac{dp}{dx} - \rho X \delta x \delta y \delta z = o$$

with similar expressions for the other pairs of faces. We have thus for the equations of fluid equilibrium,

$$\frac{dp}{dx} = \rho X$$

$$\frac{dp}{dy} = \rho Y$$

$$\frac{dp}{dz} = \rho Z$$

These three equations can, of course, be replaced by the single equation of equilibrium.

$$dp = \rho(X dx + Y dy + Z dz)$$

when dp denotes the variation of pressure corresponding to the changes dx, dy, dz, in the co-ordinates of the point at which the pressure is estimated. We see from this equation that the expression,

$$Xdx + Ydy + Zdz,$$

is either an exact differential or capable of being made so by a factor. If the forces X, Y, Z belong to a conservative system, that is, a system possessing a potential, or for which the above expression is an exact differential, we know that

$$\frac{dX}{dy} - \frac{dY}{dx} = o \text{ \&c.}$$

But when these conditions are satisfied the quantity $Xdx + Ydy + Zdz$ is an exact differential, or in the case of a system of conservative forces we have without the assistance of any integrating factor

$$Xdx + Ydy + Zdz = dR$$

when R is the potential of the forces at the point (x,y,z). It follows from this that $dp = -\rho dR$, or that p is a function of R, then for all surfaces for which R is constant p is also constant, *i.e.* the pressure is constant over all equi-potential surfaces. From these equations of equilibrium we can pass directly to the equations of motion, by means of D'Alembert's principle. Call u, v, w, the velocities of a particle of the fluid whose co-ordinates at the time t are x,y,z, thus,

$$u=\frac{dx}{dt},\ v=\frac{dy}{dt},\ w=\frac{dz}{dt};$$

let u', v', w', denote the accelerations to which these velocities give rise, then in our equations of equilibrium replacing X, Y, Z by,

$$X-u',\ Y-v',\ Z-w'$$

we have for the equations of motion

$$\frac{dp}{dx}=\rho(X-u'),$$

$$\frac{dp}{dy}=\rho(Y-v'),$$

$$\frac{dp}{xz}=\rho(Z-w')$$

when we have of course,

$$u'=\frac{du}{dt}+\frac{du}{dx}\frac{dx}{dt}+\frac{du}{dy}\frac{dy}{dt}+\frac{du}{dz}\frac{dz}{dt}$$

$$=\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)u.$$

We may for brevity replace this operator by $\frac{\mathrm{D}}{\mathrm{D}t}$, and we have thus for the equations of fluid motion the following:

$$\frac{dp}{dx}=\rho\left(\mathrm{X}-\frac{\mathrm{D}u}{\mathrm{D}t}\right),$$

$$\frac{dp}{dy}=\rho\left(\mathrm{Y}-\frac{\mathrm{D}v}{\mathrm{D}t}\right),$$

$$\frac{dp}{dz}=\rho\left(\mathrm{Z}-\frac{\mathrm{D}w}{\mathrm{D}t}\right).$$

Concerning the operator which we have denoted by $\frac{\mathrm{D}}{\mathrm{D}t}$, it is important to observe that it relates to a particular particle and not to a particular point in space; the velocities u, v, w, are functions of x, y, z and t, and denote the velocity which *any* particle has when it occupies the position denoted by x, y, z *and* $\frac{du}{dt}$

dt denotes the increase of velocity of a second particle over the one originally in this position, which arrives at this point after the lapse of time dt, while on the other hand, $\frac{\mathrm{D}u}{\mathrm{D}t}dt$ denotes the change in velocity of the original particle during this time. When the motion is very small, the terms $u\frac{d}{dx}$, &c. may be neglected, and we would have,

$$\frac{\mathrm{D}}{\mathrm{D}t}=\frac{d}{dt}.$$

To our equations of motion it is necessary to add one more, expressing the continuity of the fluid. This equation simply expresses the fact, that during any natural motion there can be neither annihilation or generation of matter, or, referring to our problem, that the amount of fluid in any space at any time must be equal to the amount originally contained in that space, increased by the amount which has entered it during the time which has been allowed to pass, diminished by the amount which has left it during that time.

Let a, b, c, denote the co-ordinates of any particle of the fluid at an initial instant, x, y, z, denote the values of these co-ordinates at the time t; now in order completely to specify the motion it is necessary to express these latter quantities as functions of initial co-ordinates and the time. Suppose further, that δa, δb, δc, are the edges of a small parallelopiped of the fluid, as these are assumed to be infinitesimal, the figure will remain a parallelopiped during the motion.

We have now for the co-ordinates of the extremities of the edges meeting in the point a, b c,

$$a+\delta a,\ b,\ c,\quad a,\ b+\delta b,\ c,\quad a,\ b,\ c+\delta c$$

at the time t the co-ordinates of these points will be,

$$\begin{array}{ccc} x, & y, & z, \\ x+\dfrac{dx}{da}\delta a, & y+\dfrac{dy}{da}\delta a, & z+\dfrac{dz}{da}\delta a \\ \vdots & \vdots & \vdots \end{array}$$

From these we arrive, by a simple geometrical process, at the volume of the parallelopiped, which is then at the time t.

$$\begin{vmatrix} \frac{dx}{da}, & \frac{dy}{da}, & \frac{dz}{da} \\ \frac{dx}{db}, & \frac{dy}{db}, & \frac{dz}{db} \\ \frac{dx}{dc}, & \frac{dy}{dc}, & \frac{dz}{dc} \end{vmatrix} \delta a \delta b \delta c$$

or representing the determinant by $\varDelta$, $\varDelta \delta a \delta b \delta c$; hence by our definition of continuity we must have

$$\varDelta = 1,$$

or in general if ρ_0 and ρ denote the initial and final densities of the fluid contained in this portion of space

$$\rho \varDelta = \rho_0$$

which simply expresses the fact that the density of the fluid contained in this portion of space must vary inversely as the volume of the space.

This equation is known as the integral equation of continuity. The form of the equation most generally employed, however, is that which expresses the fact that the rate of diminution of density bears to the density at any instant the same ratio that the rate of increase of the volume of an infinitely small portion of the fluid bears to the same infinitely small volume at the same instant. The symbolical expression of this fact constitutes the *differential equation of continuity* of the fluid.

Let the flow towards the inside of an elementary parallelopiped of the fluid be considered as positive, then the flow towards the outside will be negative. Representing as before the edges of this elementary parallelopiped by δx, δy, δz we have for the flow through the face $\delta y \delta z$ in the direction of x and during the time dt

$$\rho \delta y \delta z u dt$$

through the opposite face the flow will be during the same time

$$-\delta y \, \delta z \left(\rho u + \frac{d\rho u}{dx} . \delta x \right) dt.$$

These together give rise to an increase of mass

$$-\delta x \delta y \delta z \frac{d\rho u}{dx} dt$$

with similar expressions for the other pairs of faces respectively, perpendicular to the axis of y and z. Then the total increase of mass is

$$-\delta x \delta y \delta z \left(\frac{d.\rho u}{dx} + \frac{d.\rho v}{dy} + \frac{d.\rho w}{dz} \right);$$

but this increase of mass is also given by

$$\delta x \delta y \delta z \frac{d\rho}{dt} dt;$$

equating these values and we have for the equation of continuity

$$\frac{d\rho}{dt} + \frac{d.\rho u}{dx} + \frac{d.\rho v}{dy} + \frac{d.\rho w}{dz} = o$$

or for incompressible fluids simply,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = o$$

It may not be uninteresting to show how this differential equation of continuity can be derived from the integral equation. We have, denoting the mass of this elementary portion of fluid by m,

$$m=\rho\begin{vmatrix} \frac{dx}{da}, & \frac{dx}{db}, & \frac{dx}{dc} \\ \frac{dy}{da}, & \frac{dy}{db}, & \frac{dy}{dc} \\ \frac{dz}{\delta a}, & \frac{dz}{\delta b}, & \frac{dz}{\delta c} \end{vmatrix}\delta a\ \delta b\ \delta c$$

Differentiating this with respect to t we have

$$o=\varDelta\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\frac{d\varDelta}{dt}$$

or

$$o=\frac{\mathrm{D}\rho}{\mathrm{D}t}+\frac{\rho}{\varDelta}.\frac{d\varDelta}{dt}$$

the quantity $\frac{1}{\varDelta}\frac{d\varDelta}{dt}$ will be found by easy reductions to be equal to

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$$

and the equation thus becomes

$$o=\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$$

from which we obviously obtain the forms given above. Particular forms of this equation for special cases are often quite simple—as, for example—suppose the motion of the fluid to be wholly parallel to the plane xy; in this case we have simply

$$\frac{du}{dx}+\frac{dv}{dy}=o$$

but $\frac{du}{dx}=-\frac{dv}{dy}$ is the condition that the expression

$$udy-vdx$$

be an exact differential; calling it $d\Psi$ we have

$$u=\frac{d\Psi}{dy},\quad v=-\frac{d\Psi}{dx}.$$

The quantity Ψ is called the stream function, and all motion takes place in the direction of the curves Ψ=const. If the motion be steady, the lines Ψ =const. will form a system of tubes in the fluid, which may be called the tubes of flow. A much more general simplification of the equations of hydrodynamics exists however for certain classes of motion. It is a fact, the discovery of which is due to Lagrange that if at any time the expression

$$udx + vdy + wdz$$

is an exact differential, it will remain so throughout the motion; that is, if at any time we have

$$\frac{du}{dx} - \frac{dv}{dy}, \ \frac{dw}{dx} - \frac{du}{dz}, \ \frac{du}{dy} - \frac{dv}{dx}, = o$$

these quantities will remain so throughout the motion. Representing these quantities by ξ, η, δ, we may express this fact in another manner, viz. if at any time the motion of the fluid be irrotational, it will remain so during the entire motion. In particular, if the fluid originally at rest be set in motion, by a system of conservative forces or pressures, there will be no motion of rotation throughout the entire motion. The following proof of this theorem is that given by Sir Wm. Thomson in his paper on "Vortex Motion," Edin. Trans. 1869. As this proof does not depend upon the quantity ρ, we can give it in general representing by ω the integral $\int \frac{dp}{\rho}$. We have then from our equations of motion

$$d\omega = \mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$$
$$-\left(\frac{\mathrm{D}u}{\mathrm{D}t}dx + \frac{\mathrm{D}v}{\mathrm{D}t}dy + \frac{\mathrm{D}w}{\mathrm{D}t}dz\right)$$

Now, according to hypothesis,

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = d\mathrm{R},$$

and obviously,

$$\frac{\mathrm{D}u}{\mathrm{D}t}dx + \frac{\mathrm{D}v}{\mathrm{D}t}dy + \frac{\mathrm{D}w}{\mathrm{D}t}dz$$
$$= \frac{\mathrm{D}}{\mathrm{D}t}(udx + vdy + wdz)$$
$$-\left(u\frac{\mathrm{D}dx}{\mathrm{D}t} + v\frac{\mathrm{D}dy}{\mathrm{D}t} + w\frac{\mathrm{D}dz}{\mathrm{D}t}\right)$$

or since

$$\frac{\mathrm{D}dx}{\mathrm{D}t} = \frac{d\mathrm{D}x}{\mathrm{D}t} = du, \text{ \&c.}$$

$$d\omega = d\mathrm{R} - \frac{\mathrm{D}}{\mathrm{D}t}(udx + vdy + wdz)$$
$$+ (udu + vdv + wdw)$$

or representing $u^2 + v^2 + w^2$ by V^2,

$$\frac{\mathrm{D}}{\mathrm{D}t}udx + vdy + wdz) = d(\mathrm{R} + \tfrac{1}{2}\mathrm{V}^2 - \omega).$$

Integrating this along any arc (12) moving with the fluid we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\int(udx+vdy+wdz)=(\mathrm{R}+\tfrac{1}{2}\mathrm{V}^2-\omega)_2$$
$$-(\mathrm{R}+\tfrac{1}{2}\mathrm{V}^2-\pi);$$

If the arc be a closed circuit the second member of this equation vanishes and we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\int(udx+vdy+wdz)=o,$$

or this may be expressed by saying that *the line integral of the tangential component velocity around any closed curve of a moving fluid remains constant throughout all time.* The line integral is called the *circulation*, and the proposition may be stated. *The circulation in any closed line moving with the fluid remains constant.* In a state of rest the circulation is, of course, zero; therefore, for the assumed case of motion generated by pressures or conservative forces we have that the circulation is always zero, so that $udx+vdy+wdz$ is an exact differential. Representing this quantity by $d\varphi$ we have

$$u=\frac{d\varphi}{dx},v=\frac{d\varphi}{dy},w=\frac{d\varphi}{dz}$$

from which we have for the differential equation of continuity

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}=o$$

or simply

$$\Delta^2\varphi=o.$$

The quantity φ is appropriately called the velocity potential, and the velocity in *any* direction is expressed by the corresponding rate of change of φ. We may just here observe one fact concerning φ. If φ is a minimum at any point, *i.e.* if it increases as we go away from that point there must evidently be a positive expansion of the fluid from this point in all directions. Similarly if φ be a maximum at any point. Then the motion is in all directions towards this point and there is compression of the fluid.

If there be neither expansion nor compression of the fluid within the region bounded by a closed surface, the greatest and least values of the velocity potential in that region must be on the surface; for since there is no expansion or contraction, there can be no maximum or minimum value within this surface. If, therefore, the velocity potential is constant over the surface, it must be constant throughout the enclosed region, since its greatest and least values are now equal. In particular, if it is zero over the surface, it must be zero throughout the enclosed region. When the velocity potential exists, the equation for determining the pressure can be put into a very simple form, viz.

$$d\omega = dR - \frac{D}{Dt} d\varphi + \tfrac{1}{2} dV^2$$

integrating

$$\omega = \int \frac{dp}{\rho} = R - \frac{D\varphi}{Dt} + \tfrac{1}{2} V^2$$

but

$$\frac{D\varphi}{Dt} = \frac{d\varphi}{dt} + u^2 + v^2 + w^2 = \frac{d\varphi}{dt} + V^2$$

so that

$$\omega = \int \frac{dp}{\rho} = R - \left(\frac{d\varphi}{dt} + \tfrac{1}{2} V^2 \right)$$

or for our assumed case of incompressible fluids

$$\frac{p}{\rho}=\mathrm{R}-\left(\frac{d\varphi}{dt}+\tfrac{1}{2}\mathrm{V}_{\circ}^{2}\right)$$

Another form of the equations of fluid motion due to Lagrange is worthy of notice here, though the forms already given, or Euler's equations, are those employed in general in hydrodynamics. Since the quantities x, y, z, are functions of the initial coördinates of the point, we have

$$\begin{array}{ccccccc}\dfrac{dp}{da} & = & \dfrac{dp}{dx}\dfrac{dx}{da} & + & \dfrac{dp}{dy}\dfrac{dy}{da} & + & \dfrac{dp}{dz}\dfrac{dz}{da} \\ : & & : & & : & & :\end{array}$$

from these we have

$$\frac{dp}{dx}=\frac{1}{\Delta}\begin{vmatrix}\dfrac{dp}{da}, & \dfrac{dy}{da}, & \dfrac{dz}{da} \\ \dfrac{dp}{db}, & \dfrac{dy}{db}, & \dfrac{dz}{db} \\ \dfrac{dp}{dc}, & \dfrac{dy}{dc}, & \dfrac{dz}{dc}\end{vmatrix} \quad \&c.$$

but we have also

$$\frac{1}{\rho}\frac{dp}{dx}=\mathrm{X}-\frac{d^2x}{dt^2}, \ \&c.$$

Substituting the values of $\dfrac{dp}{dx}$ from these last equations in those giving the values of $\frac{dp}{da}$... and we have the Lagrangeian equations of fluid motion, viz.

$$\left\{\frac{d^2x}{dt^2}-\mathrm{X}\right\}\frac{dx}{da}+\left\{\frac{d^2y}{dt^2}-\mathrm{Y}\right\}\frac{dy}{dx}$$
$$+\left\{\frac{d^2z}{dt^2}-\mathrm{Z}\right\}\frac{dz}{da}+\frac{1}{\rho}\frac{dp}{da}=o$$

with two similar ones containing b and c, respectively. The reader can see that from these equations we can readily pass to the forms given before. Where the forces X, Y, Z have a potential and there is also a velocity potential, these equations become

$$\frac{d^2x}{dt^2}\frac{dx}{da}+\frac{d^2y}{dt^2}+\frac{dy}{da}\frac{d^2z}{dt^2}\frac{dz}{da}=\frac{d(\mathrm{R}-\omega)}{da}$$
$$\frac{d^2x}{dt^2}\frac{dx}{db}+\frac{d^2y}{dt^2}\frac{dy}{db}+\frac{d^2z}{dt^2}\frac{dz}{db}=\frac{d(\mathrm{R}-\omega}{db}$$
$$\frac{d^2x}{dt}\frac{dx}{dc}+\frac{d^2y}{dt^2}\frac{dy}{dc}+\frac{d^2z}{dt^2}\frac{dz}{dc}=\frac{d(\mathrm{R}-\omega)}{dc}$$

Differentiate the second of these equations with respect to c, the third with respect to b, and subtract the latter result from the former. It will be sufficient to examine only two terms of the result; we have then

$$\frac{d}{dc}\left\{\frac{d^2x}{dt^2}\,\frac{dx}{db}\right\}-\frac{d}{db}\left\{\frac{d^2x}{dt^2}\,\frac{dx}{dc}\right\}=\frac{dx}{db}\,\frac{d^2}{dt^2}\,\frac{dx}{dc}$$
$$-\frac{dx}{dc}\,\frac{d^2}{dt^2}\,\frac{dx}{db}$$

$$=\frac{d}{dt}\left\{\frac{dx}{db}\,\frac{d}{dt}\,\frac{dx}{dc}-\frac{dx}{dc}\,\frac{d}{dt}\,\frac{dx}{db}\right\}=\frac{d}{dt}\left\{\frac{dx}{db}\cdot\frac{du}{dc}\right.$$
$$\left.-\frac{dx}{dc}\,\frac{du}{db}\right\}\text{&c.,}$$

the remaining terms will be obtained by advancing the letters. We have then a quantity which differentiated for t is $=0$. Performing similar operations on the remaing pairs of equations we arrive readily at the following equations, where C_1, C_2, C_3 denote quantities which are independent of the time.

$$\left\{\frac{dx}{da}\,\frac{du}{db}-\frac{dx}{db}\,\frac{du}{da}\right\}+\left\{\frac{dy}{da}\,\frac{dv}{db}\quad\frac{dy}{db}\,\frac{dv}{da}\right\}$$
$$+\left\{\frac{dz}{da}\,\frac{dw}{db}-\frac{dz}{db}\,\frac{dw}{da}\right\}=C_1$$

$$\left\{\frac{dx}{dc}\frac{du}{da}-\frac{dx}{da}\frac{du}{dc}\right\}+\left\{\frac{dy}{dc}\frac{dv}{da}-\frac{dy}{da}\frac{dv}{dc}\right\}$$
$$+\left\{\frac{dz}{dc}\frac{dw}{da}-\frac{dz}{da}\frac{dw}{dc}\right\}=C_2,$$

$$\left\{\frac{dx}{db}\frac{du}{dc}-\frac{dx}{dc}\frac{du}{db}\right\}+\left\{\frac{dy}{db}\frac{du}{dc}-\frac{dy}{dc}\frac{du}{db}\right\}$$
$$+\left\{\frac{dz}{db}\frac{dw}{dc}-\frac{dz}{dc}\frac{dw}{db}\right\}=C_3.$$

Let now u_0, v_0, w_0 represent the values of u, v, w for $t=o$, thus at this time we have

$$u=u_0,\quad v=v_0,\quad w=w_0$$
$$x=a,\quad y=b,\quad z=c;$$

Substituting these values in the above equations and they reduce immediately to

$$\frac{du_0}{db}-\frac{dv_0}{da}=C_1$$
$$\frac{dw_0}{da}-\frac{du_0}{dc}=C_2$$
$$\frac{dv_0}{dc}-\frac{dw_0}{db}=C_3$$

We have further, since u, v, w are functions of x, y, z,

$$\frac{du}{da}=\frac{du}{dx}\frac{dx}{da}+\frac{du}{dy}\,\frac{dy}{da}+\frac{du}{dz}\,\frac{dz}{da}\,\text{\&c.}$$

If now in our determinant $\varDelta$ we denote the separate minors by A_1, B_1, $\varGamma_1$, . . . $\varGamma_3$, *i. e.*,

$$A_1=\frac{dy}{db}\,\frac{dz}{dc}-\,-\frac{dy}{dc}\,\frac{dz}{da}\text{-\&c.}$$

substituting now the values of $\frac{du}{da}$ &c. in the above equations C_1, C_2, C_3 and noting these last abreviations we have, since for incompressible fluids $\varDelta=1$,

$$A_1\left\{\frac{dv}{dz}-\frac{dw}{dy}\right\}+B_1\left\{\frac{dw}{dx}-\frac{du}{dz}\right\}$$
$$+\varGamma_1\left\{\frac{du}{dy}-\frac{dv}{dz}\right\}=\frac{dv_0}{dc}-\frac{dw_0}{db}$$

$$A_2\left\{\frac{dv}{dz}-\frac{dw}{dy}\right\}+B_2\left\{\frac{dw}{dx}-\frac{du}{dz}\right\}$$
$$+\varGamma_2\left\{\frac{du}{dy}-\frac{dv}{dz}\right\}=\frac{dw_0}{da}-\frac{du_0}{dc}$$

$$A_3\left\{\frac{dv}{dz}-\frac{dw}{dy}\right\}+B_3\left\{\frac{dw}{dx}-\frac{du}{dz}\right\}$$
$$+\varGamma_3\left\{\frac{du}{dy}-\frac{dv}{dz}\right\}=\frac{du_0}{db}-\frac{dv_0}{da}$$

Representing the quantities on the right hand side, as we appropriately may, by $2\xi_0$, $2\eta_0$, $2\zeta_0$, and solving the equations for the quantities within the parenthesis, we have

$$\xi = \xi_0 \frac{dx}{da} + \eta_0 \frac{dx}{db} + \zeta_0 \frac{dx}{dc}$$

$$\eta = \xi_0 \frac{dy}{da} + \eta_0 \frac{dy}{db} + \zeta_0 \frac{dy}{db}$$

$$\zeta = \xi_0 \frac{dz}{da} + \eta_0 \frac{db}{dz} + \zeta_0 \frac{dz}{dc}$$

If the quantities ξ_0, η_0, ζ_0, which are the initial angular velocities of the particle of the fluid whose co-ordinates at $t = o$ a, b, c, are $= o$, we have that ξ, η, ζ must also be $= o$, that is, we arrive again at the theorem that if there be no original motion of rotation in the fluid there will be none at any future time. It will be of interest to obtain the equations which were used by Helmholtz in his great memoir on vortex motion. These are simply obtained from our equations of motion. The first of these equations written out in full is

$$\frac{dw}{dx}=X-\frac{du}{dt}-u\frac{du}{dx}-v\frac{du}{dy}-w\frac{du}{dz}\ \&c.$$

Now supposing the fluid initially at rest to be set in motion by conservative forces and pressures from the exterior, the analytic conditions for this are

$$\frac{dX}{dy}-\frac{dY}{dx}=o\ \&c.$$

Therefore differentiating the first equation with respect to y and the second with respect to x, and subtracting we eliminate ω and the impressed forces, and have

$$o=\frac{d}{dx}\frac{dv}{dt}-\frac{d}{dy}\frac{du}{dt}+\frac{du}{dx}\left\{\frac{dv}{dx}-\frac{du}{dy}\right\}$$
$$+\left\{u\frac{d}{dy}\frac{dv}{dx}-u\frac{d}{dx}\frac{du}{dx}\right\}+\cdot\cdot$$

from this we have obviously

$$\frac{D}{\quad}=\frac{du}{dz}\xi+\frac{dv}{dz}\eta-\left\{\frac{du}{dx}+\frac{dv}{dy}\right\}\zeta$$

and remembering that for incompressible fluids we have

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=o$$

this becomes

$$\frac{D\zeta}{Dt}=\frac{du}{dz}\xi+\frac{dv}{dz}\eta+\frac{dw}{dz}\zeta$$

And similarly

$$\frac{D\xi}{Dt}=\frac{du}{dx}\xi+\frac{dv}{dx}\eta+\frac{dw}{dx}\zeta,$$

$$\frac{D\eta}{Dt}=\frac{du}{dy}\xi+\frac{dv}{dy}\eta+\frac{dw}{dy}\zeta.$$

The principle of the persistence of initially irrotational motion obviously follows from these equations.

§ 2.

THE POTENTIAL.

It will perhaps be as well to make a few remarks here concerning the theory of the potential. It is not the purpose in these pages to go into that subject with any degree of fullness, but as there are a few leading principles which frequently recur a brief statement and derivation of them may be of assistance to some readers.

We have already observed one fact

concerning the velocity potential, viz., that if it is constant over a closed surface containing a certain definite region that it will be constant throughout this region, and in particular, if it be $=o$ over the surface it will be $=o$ throughout the contained region. When $udx+vdy+wdz$ is an exact differential we have seen that by making it equal to $d\varphi$ we can replace the quantity

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$$

by

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}.$$

Now let σ denote any closed surface, then if ν be the outer normal to this surface we have $\frac{d\varphi}{d\nu}\,d\sigma$ for the rate of flow outwards through the element $d\sigma$ in unit of time, then the total flow outwards in time dt is equal to

$$\int\int\frac{d\varphi}{d\nu}d\sigma.dt$$

where the integration extends over the whole surface. If the space enclosed by

σ be full both at the beginning and end of this time we, of course, have

$$\int\int\frac{d\varphi}{d\nu}d\sigma=o.$$

This is the equation of continuity for the whole region. Applied to the element of volume $d\tau$ or $dx\ dy\ dz$ this gives us

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}=o \text{ or } \varDelta^2\ \varphi=o$$

If we denote by r the distance between any two points x, y, z, and a, b, c, *i. e.*

$$r^2=(x-a)^2+(y-b)^2+(z-c)^2$$

and by m a constant we see that the equation $\varDelta^2\varphi=o$ is satisfied by

$$\varphi=\frac{m}{r}$$

or $\frac{m}{r}$ is a particular solution of the partial differential equation.

The general solution is a homogeneous function of the quantities x, y, z of the degree i, where i is any positive integer. It is also well known that to every solution of the degree i then corresponds one of the degree $-(i+1)$ expressed by $\frac{\varphi i}{r^{2i+1}}$.

The expression φi is a Solid Spherical Harmonic of the degree i. The expression obtained by dividing φi by r^i which will be a function only of two quantities, viz., the angles θ and ψ, is a Spherical Surface Harmonic of the same degree.

The quantity φ is now the Potential of the mass m upon the point (x, y, z). At infinity the Potential with its derivatives vanishes, but is finite and continuous throughout the space except at the points in which the masses are found, *i. e.*, for $x=a$, $y=b$, $z=c$. Let now in the space under consideration, which is supposed to be continuously filled with masses, m denote a mass placed at a given point A and let r denote the length of a line drawn from this point to any other B—then we know that the attraction of the mass m upon the point B is given by—

$$\frac{d\varphi}{dr}=\frac{m}{r^2}.$$

Now suppose the line AB drawn to an

infinite distance, and further that the space which contains the masses m is bounded by a closed surface. The line AB will cut the surface an even number of times. Suppose now a sphere of radius unity to be described with center A, and then let the line AB describe a conical surface cutting the element $d\omega$ from the surface of the sphere, and ds_1, ds_2, &c., from the given closed surface. Let ε denote the angle between AB and the outer normal to the surface; then where the line issues from the surface cos ε will be positive and where it enters cos ε will be negative. We have now

$$ds = \frac{r^2 dw}{\cos\,\varepsilon}.$$

The normal force on the element ds is given by R cos. $\varepsilon\, ds$; but R cos. $\varepsilon\, ds$ is equal to

$$\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) dx\, dy\, dz$$

and therefore for the whole surface

$$\int\int \text{R cos. } \varepsilon\, ds = \int\int\int\left(\frac{dX}{dx}+\frac{aY}{dy}+\frac{dZ}{dz}\right)dx\, dy\, dz$$

Now since $R=\frac{m}{r^2}$ we have

$$\text{R cos. } \varepsilon\, ds = \pm\, md\omega.$$

As the point A is within the surface, the line AB first issues from the surface, giving a positive value of $md\omega$; after that alternate positive and negative values of $md\omega$ which destroy each other, so that we have simply

$$\Sigma\ \text{R cos. } \varepsilon\, d\, s = md\omega$$

and

$$\int\int \text{R cos. } \varepsilon\, d\, s = m\int\int d\omega = 4\pi m$$

Now

$$4\pi m = 4\pi\int\int\int \rho\, dx\, dy\, dz$$

$$\therefore\ \frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=4\pi\rho$$

or

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}=-4\pi\rho \text{ for } X=-\frac{d\varphi}{dx} \text{ \&c.}$$

This equation holds throughout the entire space which is filled with the masses *m*.

If we represent an element of the space under consideration by $d\tau$ and the density by ρ we may write

$$\varphi=\int\frac{\rho d\tau}{r}$$

from which

$$\frac{d\varphi}{dz}=\int\rho\frac{d\frac{1}{r}}{dz}d\tau=-\int\rho\frac{d\frac{1}{r}}{dc}d\tau$$

$$=-\int\frac{d\frac{\rho}{r}}{dc}d\tau+\int\frac{d\rho}{dc}\frac{d\tau}{r}$$

and finally

$$\frac{d\varphi}{dz}=\int ds\frac{\rho}{r}\cos.\left(\frac{n}{z}\right)+\int\frac{d\rho}{dc}\frac{d\tau}{d}$$

The second of these integrals is evidently a quantity of the same kind as φ; the first is the potential of a mass which is spread out upon the surface $\int ds$ and giving the surface density

$$\rho\cos.\left(\frac{n}{z}\right).$$

Suppose now that φ' be the potential with reference to the point (x, y, z) of a mass that is spread out upon a surface giving the surface density μ; we have

$$\varphi' = \int \frac{\mu d\sigma}{r}.$$

Assume the rectangular axes so that z is normal to the surface; assume on z a point infinitely near the surface, and suppose a circular cylinder with radius P′ and having the axis of z for its axis of figure to cut the surface; suppose P′ indefinitely small but infinitely large with respect to the ordinate z. Let φ_1' denote the portion of φ belonging to that part of surface included in the cylinder; the remaining portion $\varphi' - \varphi_1'$ will not become infinite or discontinuous by z becoming either $=o$ or passing through zero.

$$\varphi_1' = 2\pi\mu \int_o^{\mathrm{P}'} \frac{\rho' d\rho'}{\sqrt{\rho'^2 + z^2}}$$

$$\varphi_1' = 2\pi\mu[\sqrt{\mathrm{P}'^2 + z^2} - \sqrt{z^2}]$$

By neglecting infinitesimals we have

$$\varphi_1' = o$$

or φ' remains finite and continuous if the point under consideration passes through the surface, *i.e.*, if $\sqrt{z^2} = \pm z$. Further.

$$\frac{d\varphi_1'}{dz} = 2\pi\mu \left\{ \frac{z}{\sqrt{P'^2 + z^2}} - \frac{z}{\sqrt{z^2}} \right\}$$

or since, with respect to z, P' is infinitely great,

$$\frac{d\varphi_1'}{dz} = -2\pi\mu \frac{z}{\sqrt{z^2}}$$

or if z be positive

$$\frac{d\varphi_1'}{dz} = -2\pi\mu$$

if z be negative

$$\frac{d\varphi_1'}{dz} = +2\pi\mu.$$

But $\varphi' - \varphi_1'$ is continuous and also $\frac{d(\varphi' - \varphi_1')}{dz}$; consequently when z changes from positive to negative passing through zero $\frac{d\varphi'}{dz}$ changes suddenly by the amount $-4\pi\mu$. Now as we have taken z in the direction of the normal this fact can be expressed as follows: Call the inner normal ν_1 and the outer normal ν_2 then we have

$$\frac{d\varphi'}{d\nu_1}+\frac{d\varphi'}{d\nu_2}=-4\pi\mu$$

which may be called the *characteristic equation* of φ' at the surface.*

Suppose now that we have a surface over which a mass is so distributed as to give rise to the surface density that we have denoted by μ giving then the potential φ'. We will use, for the present, the symbol φ to express a general function, which satisfies the equation,

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}=o$$

And we will take now U to represent the potential in the space under consideration, V for what we have denoted by φ', the potential of a surface over which a mass is distributed, as mentioned above. Now we will introduce a new quantity W, which we proceed to define: At every point of the surface that we have spoken of conceive normals (positive) to be drawn, lay off on these infinitely small lengths, and through these points conceive another surface to pass, the elements of which correspond to the elements of the first surface and consider that on each element of this new surface a mass is distributed equal to that on the corresponding element of the first surface but of opposite sign. Represent by k the negative product of the density of the mass on the element $d\sigma$ of the first surface, by $d\sigma$; then if (a, b, c) denote the element $d\sigma$ and W is the potential of this element at the point (x, y, z); we have

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* **Maxwell, Elec. and Mag., p. 82, Vol. I.**

$$W = \int k \frac{d\frac{1}{r}}{d\nu}\, d\sigma$$

Since, $\frac{d\frac{1}{r}}{d\nu} = \frac{d\frac{1}{r}}{dr}\frac{dr}{d\nu}$

we have

$$W = -\int \frac{k d\sigma}{r^2} \cos. (r\nu)$$

We can also express this in another way. Conceive a sphere of unit radius described about $(x, y, z,)$ as center; also conceive acone on $d\sigma$ as base and having (x, y, z) as vertex, to cut the sphere, the area of the included portion being $d\Sigma$, then

$$\frac{d\frac{1}{r}}{d\nu}d\sigma = \pm d\Sigma$$

the upper or lower sign to be taken according as cos. $(r\nu)$ is positive or negative. The cosine can only change its sign by $(r\nu)$ passing through $\frac{\pi}{2}$; evidently this is the case only when the point can lie on a tangent to the surface; then supposing that cos. $(r\nu)$ does not change its sign, we have

$$W = \int \mp k d\Sigma$$

when the upper or lower sign is to be taken according as cos. $(r\nu)$ is positive or negative. If the above condition is not fulfilled, the surface may be divided into parts so that each part can satisfy the imposed conditions; then will W be given as the sum of the corresponding expressions for each part. In order to examine whether or not discontinuity occurs in the value of W, by the point to which it refers approaching indefinitely near the surface —coinciding with it or passing through it—we will choose the axis so that z is normal to the surface, and assume a point on z indefinitely near the surface; now by exactly the same process as that before employed, we see that for a negative z we have

$$W_1 = -2\pi k$$

and for a positive z

$$W_1 = +2\pi k$$

W_1 corresponding to the small portion cut out of the surface by a circular cylinder of indefinitely small radius P, which is nevertheless infinitely large as regards z. Now since W_1 is independent of z, W cannot become infinitely great by z becoming infinitely small; and since W_1 suddenly changes by $4\pi k$, W does so likewise. From our equation for W, we have for W_1, since it refers to an indefinitely small portion of the surface for which k is constant,

$$W_1 = -k\int \frac{d\frac{1}{r}}{dz} d\sigma = 2\pi k \int_o^{\mathrm{P}} \frac{z\rho d\rho}{(\rho^2+z^2)^{\frac{3}{2}}}$$

since

$$r = \sqrt{\rho^2+z^2} \text{ and } \frac{d\frac{1}{r}}{dz} = \frac{d\frac{1}{r}}{dr}\,\frac{dr}{dz} = -\frac{1}{r^2}\,\frac{dr}{dz},$$

or

$$W_1 = 2\pi k \left\{ \frac{z}{\sqrt{z^2}} - \frac{z}{\sqrt{P^2 + z^2}} \right\}$$

from this follows

$$\frac{dW_1}{dz} = -2\pi k \frac{P^2}{(P^2 + z^2)^{\frac{3}{2}}}$$

or, since z^2 is negligeable with respect to P^2,

$$\frac{dW_1}{dz} = -2\pi k \frac{1}{P}$$

The second member being independent of z, $\frac{dW'}{dz}$ for $z = o$ is finite and continuous. Thus we see that the potential W for this *double layer* is finite always, but changes suddenly by the amount $4\pi k$ on the point to which it refers passing through the surface in the direction of ν; the quantity $\frac{dW}{d\nu}$ is, however, finite and continuous. With a perfectly arbitrary co-ordinate system the quantities $\frac{dW}{dx}$, $\frac{dW}{dy}$, $\frac{dW}{dz}$, will in general suffer discontinuity, since k is in general not constant over all the surface; if k be constant the differential co-efficients will suffer no discontinuity at the surface.

Suppose that we have two functions U and V of x, y, z, which with their derivations are single valued and continuous in the space under consideration, which is bounded by a closed surface. We have the identical equations

$$\frac{dU}{dx}\frac{dV}{dx}+U\frac{d^2V}{dx^2}=\frac{d}{dx}\left(U\frac{dV}{dx}\right),$$
$$\frac{dU}{dy}\frac{dV}{dy}+U\frac{d^2V}{dy^2}=\frac{d}{dy}\left(U\frac{dV}{dy}\right),$$
$$\frac{dU}{dz}\frac{dV}{dz}+U\frac{d^2V}{dz^2}=\frac{d}{dz}\left(U\frac{dV}{dz}\right).$$

Add and multiply by the element of the space $d\tau$, we have by changing in the second member a volume into a surface integral,

$$\text{(A)}\int\left(\frac{dU}{dx}\frac{dV}{dx}+\frac{dU}{dy}\frac{dV}{dy}+\frac{dU}{dz}\frac{dV}{dz}\right)d\tau$$
$$=-\int d\tau\, U\varDelta^2V-\int d\sigma U\frac{dV}{d\nu}.$$

This equation expresses what is known as Green's Theorem. By an interchange of U and V, which from the nature of the functions can be effected, the first member of this equation will not change. We will have then

$$(B) \int d\sigma\left(U\frac{dV}{d\nu}+V\frac{dU}{d\nu}\right) = \int d\tau\,(V\Delta^2U-U\Delta^2V)$$

If U and V are velocity potentials this gives,

$$\int d\sigma\left(U\frac{dV}{d\nu}-V\frac{Ud}{d\nu}\right)=o$$

In the preceding equation (a) suppose U=V, then it gives,

$$\int d\tau\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right\} = -\int d\sigma V\frac{dV}{d\nu}$$

or

$$2T=-\int V\left\{\frac{dV}{dx}\cos.(\nu x)+\frac{dV}{dy}\cos.(\nu y) + \frac{dV}{dz}\cos.(\nu z)\right\}d\sigma$$

An expression for the energy in the form of a surface integral.

"The irrotational motion of incompressible fluid in a simply connected closed space Σ is completely determined by the normal velocities over the surface Σ. If Σ be a material envelope, it is evident that an arbitrary normal velocity may be impressed upon its surface, which normal velocity must be shared by the fluid immediately in contact, provided that the whole volume inclosed remain unaltered. If the fluid be previously at rest, it can acquire no molecular rotation under the operation of the fluid pressures, which shows that it must be possible to determine a function φ, such that $\Delta^2\varphi=o$ throughout the space inclosed by Σ, while over the surface $\frac{d\varphi}{d\nu}$ has a prescribed value limited only by the condition

$$\int\int\frac{d\varphi}{d\nu}d\sigma=o$$

"By Green's theorem if $\Delta^2\varphi=o$,

$$\int\int\int\left\{\left(\frac{d\varphi}{dx}\right)^2+\left(\frac{d\varphi}{dy}\right)^2+\left(\frac{d\varphi}{dz}\right)^2\right\}d\tau=\int\int\varphi\frac{d\varphi}{d\nu}d\sigma$$

the integration on the right hand side extending over the surface Σ, that on the left hand side over the volume. Now if φ and $\varphi + \Delta\varphi$ be two functions, satisfying Laplace's equation, and giving prescribed values of $\frac{d\varphi}{d\nu}$, then the difference, $\Delta\varphi$ is a function also satisfying Laplace's equation, and making $\frac{d\Delta\varphi}{d\nu}$ vanish over the surface of Σ. Under these circumstances the surface integral in the preceding equation vanishes, and we infer that at every point of Σ, $\frac{d\Delta\varphi}{dx}$, $\frac{d\Delta\varphi}{dy}$, $\frac{d\Delta\varphi}{dz}$ must be equal to zero. In other words, $\Delta\varphi$ must be constant and the two motions identical. As a particular case, there can be no motion of the irrotational kind within the volume Σ, independently of a motion of the surface."

The line described by a point in the fluid, moving always in the direction of the resultant velocity, is, as has been mentioned for a simple case, called a stream line. If φ denote the velocity potential, we have obviously for the differential equations of a stream line,

$$\frac{dx}{\frac{d\varphi}{dx}}=\frac{dy}{\frac{d\varphi}{dy}}=\frac{dz}{\frac{d\varphi}{dz}}$$

These lines evidently cut at right angles, the surface $\varphi=$const.

If the normal velocity at every point of this surface is equal to zero, we must have

$$\frac{d\varphi}{d\nu}=o$$

and this equation will represent a surface which cuts at right angles the surface $\varphi=$const., and is consequently made up of stream lines; such a surface is appropriately termed a surface of flow. The following properties of such surfaces, though not directly bearing upon the matter in hand, will possibly be of interest:

Suppose that we have for the equation of a family of surfaces

$$f(x, y, z)=q,$$

q being a variable parameter by giving constant values to which we obtain the equation of each member of the family. Make

$$\left(\frac{dq}{dx}\right)^2+\left(\frac{dq}{dy}\right)^2+\left(\frac{dq}{dz}\right)^2=\frac{1}{\nu^2}$$

Then we have for the direction cosines of the normal in the direction in which q increases

$$\cos.\,(\nu x)=\nu\frac{dq}{dx},\cos\,(\nu y)=\nu\frac{dq}{dy},$$

$$\cos.(\nu z)=\nu\frac{dq}{dz}$$

if N be the component of the flow normal to the surface, u, v, w being the components parallel to x, y and z respectively, we have

$$\mathrm{N}=\nu\left(u\frac{dq}{dx}+v\frac{dq}{dy}+w\frac{dq}{dz}\right).$$

Hence, if N be zero, there will be no flow through the surface, which may then be called a surface of flow; we have then for the equation of such a surface,

$$u\frac{dq}{dx}+v\,\frac{dq}{dy}+w\,\frac{dq}{dz}=o.$$

If there be another family of surfaces whose parameter is q', and these are surfaces of flow, then

$$u\frac{dq'}{dx}+v\frac{dq'}{dy}+w\frac{dq'}{dz}=o$$

If we have still a third family of surfaces whose parameter is q'' that are surfaces of flow, then again,

$$u\frac{dq''}{dx}+v\frac{dq''}{dy}+v\frac{dq''}{dz}=o$$

eliminating u, v and w between these, and we obtain

$$\begin{vmatrix} \frac{dq}{dx}, & \frac{dq}{dy}, & \frac{dq}{dz} \\ \frac{dq'}{dx}, & \frac{dq'}{dy}, & \frac{dq'}{dz} \\ \frac{dq''}{dx}, & \frac{dq''}{dy}, & \frac{dq''}{dz} \end{vmatrix}=o$$

which is only satisfied by making $q''=$ some function of q and q'. Suppose that we have only the two first of these equations viz.:

$$u\frac{dq}{dx}+v\frac{dq}{dy}+w\frac{dq}{dz}=o$$

$$u\frac{dq'}{dx}+v\frac{dq'}{dy}+w\frac{dq'}{dz}=o$$

By eliminating u, v, w in turn from these equations we arrive at the following where Φ is an undetermined function of q and q',

$$u=\Phi\left(\frac{dq}{dy}\frac{dq'}{dz}-\frac{dq}{dz}\frac{dq'}{dy}\right)$$

$$v=\Phi\left(\frac{dq}{dz}\frac{dq'}{dx}-\frac{dq}{dx}\frac{dq'}{dz}\right)$$

$$w=\Phi\left(\frac{dq}{dx}\frac{dq'}{dy}-\frac{dq}{dy}\frac{dq'}{dx}\right)$$

When one of the functions represented by q and q' is known, it is possible so to determine the other, that Φ shall be = unity. The flow in the direction of the normals to these surfaces being $=o$, this flow can only take place along the surface, and the intersection of the two surfaces will be a line of flow or stream line. A tube of flow or a stream filament, is a tube whose bounding surfaces are made up of lines of flow. If the two parameters q and q' have a series of values given to them, they will form a double system of surfaces dividing space up into a number of tubes, each of which will be a tube of flow. We will go further into the consideration of stream lines in another place.

§ 3.

PLANE WAVES.

Having thus briefly stated some of the more simple of the general properties of the equations of fluid motion we will now proceed to examine some of the problems which present themselves most naturally to the student. The first case that we shall take up is that of the motion of water in *plane waves* when the excursions of each particle are very small.

When a body of water originally in a state of rest is endowed with a wave motion each particle of the mass has a motion of oscillation or, describes a closed curve in such a manner as to cause the particle of water, after a certain definite lapse of time, to resume its original position on the surface of the wave. By wave, is to be of course, understood simply the *forms* which the water assumes under the action of the disturbing force. Plane waves are those in which the motion of every particle is parallel to a certain fixed plane, and they may be generated by bringing a solid body, *e.g.*, a cylinder, in contact with the surface of the water contained in a rectangular canal of uniform depth, and in such a manner that the line of contact shall be at right angles to the length of the canal. Plane waves will be generated the instant the contact takes place; these will travel along the entire length of the canal; impinge on the ends and return; we will in our problem, however, limit ourselves at first to the case of a canal of indefinite length and then need not take account of the phenomena at the ends. Our mass of water being then supposed, contained in a canal as described, we will now proceed to the mathematical examination of the waves generated by such a disturbing force as has been mentioned.

Assume the axis of z vertical and positive downwards, the axis of x parallel to the length of the canal, and that of y at right angles to its sides; the origin being on the surface of the water at rest. Let now x_0, y_0, z_0 denote the initial values of the co-ordinates of a particle and let u, v, w denote the displacements which the particle undergoes in the directions of x, y, and z respectively. For plane waves advancing in the direction of x, we have of course $v=o$, and the equation of continuity assumes the form

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dz^2}=o$$

the values of u and w being

$$u=\frac{d\varphi}{dx}, \quad w=\frac{d\varphi}{dz}.$$

We have now to express φ as a function of x, z, and t, and, from the nature of the motion, periodic with respect to the last.

The simplest way of expressing this periodicity, will be by introducing in φ a factor which shall be a trigonometric function of the time. We may assume φ in the form

$$\varphi=\varepsilon^{\pm\sigma z}f(x)\left.\begin{matrix}\sin.\\ \cos.\end{matrix}\right\}F(t)$$

when σ is a constant, and the forms of f and F are to be determined, the latter is easily obtained by the following considerations. We know that φ must not change if we increase t by the time of oscillation, denoting this by τ, and φ becomes

$$=\varepsilon^{\pm\sigma z}f(x)\left.\begin{matrix}\sin.\\ \cos.\end{matrix}\right\}F(t+\tau)$$

Now in order that φ may be a periodic function of t, we must have

$$F(t+\tau)=F(t)+2\pi$$

which gives by expansion

$$\tau F'(t)+\frac{\tau^2}{1.2}F''(t)+\ .\ .\ .=2\pi$$

from which since τ is constant,

$$F''=F'''=,\ \&c.\ .\ .=o$$

$$\therefore\qquad F(t)=\int_o^t F'(t)dt\frac{2\pi}{\tau}t$$

and φ now assumes the form

$$\varphi=\varepsilon^{\pm\sigma z}f(x)\begin{matrix}\sin.\\ \cos.\end{matrix}\frac{2\pi}{\tau}t.$$

Now for the determination of f; substituting this value of φ in the differential equation of continuity $\Delta^2\varphi=o$ and it becomes

$$\frac{d^2f}{dx^2}+\sigma^2f=o$$

Integrating this and we have for f the equation

$$f=\mathrm{A}\ \sin.\sigma x+\mathrm{B}\cos.\sigma x$$

when A and B are the constants of integration. This gives us now

$$\varphi=\varepsilon^{\pm\sigma z}\left(\mathrm{A}\ \sin.\sigma x+\mathrm{B}\cos.\sigma x\right)\begin{matrix}\sin.\\ \cos.\end{matrix}\frac{2\pi}{\tau}t.$$

This obviously may be written in this form

$$\varphi=$$

$$\left\{\begin{aligned}&\varepsilon^{\sigma z}\left(a_1\sin.\sigma x\sin.\frac{2\pi}{\tau}t+b_2\cos.\sigma x\cos.\frac{2\pi}{\tau}t\right)\\ +&\varepsilon^{\sigma z}\left(b_1\cos.\sigma x\sin.\frac{2\pi}{\tau}t+a_2\sin.\sigma x\cos.\frac{2\pi}{\tau}t\right)\\ +&\varepsilon^{-\sigma z}\left(a_1\sin.\sigma x\sin.\frac{2\pi}{\tau}t+\beta_2\cos.\sigma x\cos.\frac{2\pi}{\tau}t\right)\\ +&\varepsilon^{-\sigma z}\left(\beta_1\cos.\sigma x\sin.\frac{2\pi}{\tau}t+a_2\sin.\sigma x\cos.\frac{2\pi}{\tau}t\right)\end{aligned}\right.$$

By supposing the constants a, b, ... positive we can, by making the proper ones vanish and establishing certain relations among the remaining ones, obtain an expression for φ which shall contain as a factor the sine of the sum or the cosine of the difference of the quantities $\frac{2\pi}{\tau}t$ and σx. It is of course desirable to introduce the quantity x into the trigonometric factor as the form of φ is alike unaltered if we increase t by the time of oscillation or x by the wave length. Then making

$$a_1=b_2=\alpha_1=\beta_2=o$$

and for a simple advancing wave making,

$$b_1=a_2=\alpha_1,\ \beta_1=\alpha_2=a_2$$

we have

$$\varphi=\left(a_1\varepsilon^{\sigma z}+a_2\varepsilon^{-\sigma z}\right)\sin.\left(\frac{2\pi}{\tau}t+\sigma x\right).$$

We have here before spoken of φ as the velocity function—there is a manifest appropriateness in this case in calling it the *wave function*—a name that we shall adopt for the present.

The quantity

$$\left(a_1\varepsilon^{\sigma z}+a_2\varepsilon^{-\sigma z}\right)$$

is called the *amplitude* of the wave, and is evidently the maximum value of φ; τ is the *periodic time*, or *period*, after the lapse of which the values of φ recur; and σx determines the *phase* of the wave at the moment from which t is measured. It is evident that if we have any number of wave functions φ', φ'' . . . which satisfy the differential equations

$$\Delta^2\varphi'=o, \quad \Delta^2\varphi''=o, \text{ \&c.}$$

that this sum must also satisfy the equation

$$\Delta^2\Sigma\varphi=o$$

or any number of wave functions may be compounded into one resultant by simple addition.

Before proceeding to the general problem we will examine the simple case of only one wave function. From the value given above for φ we have

$$u=\sigma\left(a_1\varepsilon^{\sigma z}+a_2\varepsilon^{-\sigma z}\right)\cos.\left(\frac{2\pi}{\tau}t+\sigma x\right)$$

$$w=\sigma\left(a_1\varepsilon^{\sigma z}-a_2\varepsilon^{-\sigma z}\right)\sin.\left(\frac{2\pi}{\tau}t+\sigma x\right)$$

From these we see at once that the displacements u and w satisfy the equation of an ellipse whose semi axes are

$$\sigma\left(a_1\varepsilon^{\sigma z}+a_2\varepsilon^{-\sigma z}\right)$$
$$\sigma\left(a_1\varepsilon^{\sigma z}-a_2\varepsilon^{-\sigma z}\right).$$

These values of φ, u and w can however, be further simplified by finding what relation exists between the quantities a_1 and a_2. To find this relation, we proceed to examine the forces which act on any particle of the fluid; these are well known to be of the form

$$-\frac{d^2u}{dt^2},\ o,\ -\frac{d^2w}{dt^2}+g,$$

and the elementary equations of motion thus become

$$\frac{1}{\rho}\frac{dp}{dx}=-\frac{d^2u}{dx^2}$$
$$\frac{1}{\rho}\frac{dp}{dy}=-\frac{d^2w}{dt^2}+g$$

from these we have

$$p=\rho\left\{\int -\frac{d^2u}{dt^2}\,dx+\int -\frac{d^2w}{dt^2}dz\right.$$
$$\left.+\int gdz\right\}+\text{const.};$$

but

$$-\frac{d^2u}{dt^2}=-\frac{d}{dx}\frac{d^2\varphi}{dt^2}=\left(\frac{2\pi}{\tau}\right)^2\frac{d\varphi}{dx},$$
$$-\frac{d^2w}{dt^2}=-\frac{d}{dz}\frac{d^2\varphi}{dt^2}=\left(\frac{2\pi}{\tau}\right)^2\frac{d\varphi}{dz},$$

therefore,

$$p=\rho\left\{\left(\frac{2\pi}{\tau}\right)^2\varphi+gz\right\}+\text{const.}$$

The constant is evidently p_0 the initial pressure, the first part of the second number of this equation being the increase of pressure at the time t above what it was at the initial instant.

As we suppose ourselves limited to the case of very small motions, we may regard u and w as quantities of the first order; then it is obvious from the form of the expressions obtained for these quantities that σ is of the same order—the other factors being in general of finite magnitude. Then any terms which contain σu, or σw being quantities of the second order, may, with reference to those of the first order of magnitude, be discarded.

These considerations now enable us to determine p wholly in terms of the initial co-ordinates. To do this it is only necessary to write $x=x_0+u$, $z=z_0+w$, then from what has just been said we have at once

$$\sigma x=\sigma x_0,$$
$$\sigma z=\sigma z_0,$$

Substituting these in the expression for φ and this again in the equation giving the pressure and we obtain for the latter

$$p=\rho\left\{\left(\frac{2\pi}{\tau}\right)^2\left(a_1\varepsilon^{\sigma z_0}+a_2\varepsilon^{-\sigma z_0}\right)\sin.\left(\frac{2\pi}{\tau}t+\sigma x_0\right)+gz_0+gw\right\}+p_0$$

or as it may be written substituting for w its value

$$p=\rho\left\{\left(\frac{2\pi}{\tau}\right)^2\left(a_1\varepsilon^{\sigma z_0}+a_2\varepsilon^{-\sigma z_0}\right)\right.$$
$$\left.+g\sigma\left(a_1\varepsilon^{\sigma z_0}-a_2\varepsilon^{-\sigma z_0}\right)\right\}\sin.\left(\frac{2\pi}{\tau}t+\sigma x\right)$$
$$+p_0+\rho g z_0$$

Now for the determination of the constants, we observe first: that particles of the fluid originally on the bottom of the canal, must necessarily remain there during the motion; second, for particles of the fluid whose z co-ordidates are equal to zero, *i.e.*, for particles on the surface of the fluid at rest, p must be a constant $=p_0$. Let h denote the depth of the canal and the first of these conditions is evidently reached by making $w=o$ for those particles for which $z=h$. This gives us then

$$w=\sigma\left(a_1\varepsilon^{\sigma h}-a_2\varepsilon^{-\sigma h}\right)\sin.\left(\frac{2\pi}{\tau}t+\sigma x\right)=o$$

and consequently

$$a_1\varepsilon^{\sigma h}-a_2\varepsilon^{-\sigma h}=o$$

from which follows,

$$\frac{a_1}{a_2}=\frac{\varepsilon^{-\sigma h}}{\varepsilon^{\sigma h}}$$

Again, make $z=o$ and $p=p_0$; this gives obviously,

$$\left(\frac{2\pi}{\tau}\right)^2(a_1+a_2)+g\sigma(a_1-a_2)=o$$

from which

$$\left(\frac{2\pi}{\tau}\right)^2=-g\sigma\frac{a_1-a_1}{a_1+a_2}=g\sigma\frac{\varepsilon^{\sigma h}-\varepsilon^{-\sigma h}}{\varepsilon^{\sigma h}+\varepsilon^{-\sigma h}}$$

Now making for brevity

$$\sigma a_2\varepsilon^{-\sigma h}=-a$$

And our expressions for u and w become,

$$u=-a\left(\varepsilon^{-\sigma(h-z)}+\varepsilon^{\sigma(h-z)}\right)\cos.\left(\frac{2\pi}{\tau}t+\sigma x\right)$$

$$w=-a\left(\varepsilon^{-\sigma(h-z)}-\varepsilon^{\sigma(h-z)}\right)\sin.\left(\frac{2\pi}{\tau}t+\sigma x\right)$$

By introducing the wave length now we can determine the constant σ. Reverting to the expression previously given for φ, and for convenience retain the *two* constants a_1 and a_2—this was

$$\varphi=\left(a_1\varepsilon^{\sigma z}+a_2\varepsilon^{-\sigma z}\right)\sin.\left(\frac{2\pi}{\tau}t+\sigma x\right).$$

Let l denote the wave length; we know that φ will remain unchanged by writing for x the quantity $x+l$, as this simply has the effect of transferring the origin of co-ordinates from one end of the wave to the other. This substitution gives

$$\varphi=\left(a_1\varepsilon^{\sigma z}+a_2\varepsilon^{-\sigma z}\right)\sin.\left(\frac{2\pi}{\tau}t+\sigma[x+l]\right).$$

In order that φ remain unchanged, we must clearly have

$$\sigma=\frac{2\pi}{l}.$$

If we call ω the velocity of each particle, we have also

$$l=\tau\omega,\text{ or }\tau=\frac{l}{\omega}$$

Substituting those in the expression obtained above for p, and it becomes

$$p=p_0+\rho g z_0+\rho\left\{g\sigma\frac{\varepsilon^{\sigma h}-\varepsilon^{-\sigma h}}{\varepsilon^{\sigma h}+\varepsilon^{-\sigma h}}\left(a_2\frac{\varepsilon^{-\sigma h}}{\varepsilon^{\sigma h}}\varepsilon^{\sigma z_0}+a_2\,\varepsilon^{-\sigma z_0}\right)\right.$$

$$\left.+g\sigma\left(a_2\frac{\varepsilon^{-\sigma h}}{\varepsilon^{\sigma h}}\varepsilon^{\sigma z_0}-a_2\varepsilon^{-\sigma z_0}\right)\right\}$$

$$\text{sin. }\frac{2\pi}{\rho}(\omega t+x_0)$$

which reduces easily to

$$p=p_0+\rho g z^0-z\rho g a\left\{\frac{\varepsilon^{\frac{2\pi}{l}z_0}-\varepsilon^{-\frac{2\pi}{l}z_0}}{\varepsilon^{\frac{2\pi}{l}h}+\varepsilon^{-\frac{2\pi}{l}h}}\right\}\text{ sin. }$$

$$\frac{2\pi}{l}\left(\omega t+\varepsilon\right)$$

The same substitutions give us for φ the value

$$\varphi=-\frac{al}{2\pi}\left\{\varepsilon^{\frac{2\pi}{l}(h-z)}+\varepsilon^{-\frac{2\pi}{l}(h-z)}\right\}\sin.\frac{2\pi}{l}(\omega t+x)$$

and in like manner we can obtain for u and w the values

$$u=-a\left\{\varepsilon^{-\frac{2\pi}{l}(h-z)}+\varepsilon^{\frac{2\pi}{l}(h-z)}\right\}\cos.\frac{2\pi}{l}(\omega t+x),$$

$$w=-a\left\{\varepsilon^{-\frac{2\pi}{l}(h-z)}-\varepsilon^{\frac{2\pi}{l}(h-z)}\right\}\sin.\frac{2\pi}{l}(\omega t+x).$$

The value of ω is easily obtained.

$$\left(\frac{2\pi}{\tau}\right)^2=\left(\frac{2\pi\omega}{l}\right)^2=g\sigma\frac{\varepsilon^{\sigma h}-\varepsilon^{-\sigma h}}{\varepsilon^{\sigma h}+\varepsilon^{-\sigma h}}=g\frac{2\pi}{l}\,\frac{\varepsilon^{\frac{2\pi}{l}h}-\varepsilon^{-\frac{2\pi}{l}h}}{\varepsilon^{\frac{2\pi}{l}h}+\varepsilon^{-\frac{2\pi}{l}h}}$$

from this is readily obtained

$$\omega=\sqrt{\frac{gl}{2\pi}\cdot\frac{\varepsilon^{\frac{2\pi}{l}h}-\varepsilon^{-\frac{2\pi}{l}h}}{\varepsilon^{\frac{2\pi}{l}h}+\varepsilon^{-\frac{2\pi}{l}h}}}$$

In the discussion of these values for φ, u, w and p lies the whole theory of the motion of plane waves in a perfect fluid. We will now proceed to an examination of these quantities. Denote by z' the vertical ordinate at the time t of a particle on the wave surface whose other co-ordinates are x and y.

$$w=\frac{dz'}{dt}=-a\left\{\varepsilon^{-\frac{2\pi}{l}(h-z)}-\varepsilon^{\frac{2\pi}{l}(h-z)}\right\}\sin.\frac{2\pi}{l}(\omega t+x)$$

integrating

$$z'=\frac{a\rho}{2\pi\omega}\left\{\varepsilon^{\frac{2\pi}{l}(hz)}-\varepsilon^{\frac{2\pi}{l}(h-z)}\right\}\cos.\frac{2\pi}{l}(\omega t+x).$$

Differentiating this expression with respect to x and we obtain.

$$\frac{dz'}{dx}=-\frac{a}{w}\left\{\varepsilon^{-\frac{z\pi}{l}(h-z)}-\varepsilon^{\frac{z\pi}{l}(h-z)}\right\}\sin.\frac{2\pi}{l}(\omega t+x)$$

This vanishes for the values

$$x=o, \quad t=o$$
$$x=\tfrac{l}{2}, \quad t=\tfrac{\tau}{2}$$
$$x=l, \quad t=\tau$$

which three are the values of x and t for the points of maxima and minima of the longitudinal section of the wave. Differentiating again

$$\frac{d^2z'}{dx^2}=-\frac{a\rho}{2\pi\omega}\left\{\varepsilon^{-\frac{2\pi}{l}(h-z)}-\varepsilon^{\frac{2\pi}{l}(h-z)}\right\}\cos.\ \frac{2\pi}{l}(\omega t+x)$$

This vanishes for $x=\frac{l}{4}$, and $\frac{3l}{4}$, with $t=\frac{\tau}{4}$, and $\frac{3\tau}{4}$. Consequently there are points of contra-flexure at $\frac{1}{4}$ and $\frac{3}{4}$ of the wave length. It is obvious, from these considerations, what the curve is.

It has before been remarked that u and w satisfy the equation of an ellipse; this is now

$$\frac{u^2}{a^2\left\{\varepsilon^{-\frac{2\pi}{l}(h-z)}+\varepsilon^{\frac{2\pi}{l}(h-z)}\right\}^2}+\frac{u^2}{a^2\left\{\varepsilon^{-\frac{2\pi}{l}(h-z)}-\varepsilon^{\frac{2\pi}{l}(h-z)}\right\}^2}=1$$

The plane of the ellipse is vertical and its longer axis is in the direction of the motion of the wave. Suppose now the particle under consideration to lie very near the surface of the wave—that is, z is very small as compared with h; then the terms containing $\varepsilon^{-\frac{2\pi}{l}(h-z)}$ may obviously be discarded, and the only other terms which remain in the expressions for the semi-axes of the ellipse will depend on

$$\varepsilon^{\frac{2\pi}{l}(h-z)}=\varepsilon^{\frac{2\pi}{l}h}\,.\,\varepsilon^{-\frac{2\pi}{l}z}=\mathrm{A}\varepsilon^{-\frac{2\pi'}{l}z}.$$

The equation of the ellipse thus becomes

$$\frac{u^2}{A^2\varepsilon^{-\frac{4\pi}{l}z}}+\frac{v^2}{A^2\varepsilon^{-\frac{4\pi}{l}z}}=1$$

The equation of a circle whose radius is

$$A\varepsilon^{\frac{2\pi}{l}z}=R$$

Of course the same result would be obtained by supposing the depth of the fluid infinite. Thus for particles near the surface of a body of water of finite depth—or for particles anywhere within the mass of a body of water of infinite depth—the motion is in a vertical circle whose radius is given above. Suppose again that the wave has an appreciable length—say $l=h$; then for particles very near the surface the semi-axes become very nearly.

$$\varepsilon^{2\pi}+\varepsilon^{-2\pi}, \text{ and } \varepsilon^{2\pi}-\varepsilon^{-2\pi}$$

or the path of the particle is nearly circular—the ratio between these quantities being nearly 1.000,007.

The lengths of the axis continuously decrease as z increases. This is obvious in the case of the vertical axis given by

$$2a\left\{\varepsilon^{\frac{2\pi}{l}(h-z)} - \varepsilon^{-\frac{2\pi}{l}(h-z)}\right\}$$

for as z becomes larger the exponents in this quantity become smaller, thus causing the first term in the brackets to diminish as the second increases, and consequently making the total value of the quantity diminish rapidly. Take now the horizontal axis denoted by

$$2a\left\{\varepsilon^{\frac{2\pi}{l}(h-z)} + \varepsilon^{-\frac{2\pi}{l}(h-z)}\right\}=\beta$$

differentiating this with respect to z and we have

$$\frac{d\beta}{dz}=2a\frac{2\pi}{}\left\{\varepsilon^{-\frac{2\pi}{l}(h-z)} - \varepsilon^{\frac{2\pi}{l}(h-z)}\right\}$$

For $z<h$ the second of these terms is always greater than the first and, consequently, $\frac{d\beta}{dz}$ is negative; but the incre-

ment dz is supposed positive, consequently $d\beta$ is negative, or the axis decreases as z increases, *i.e.*, as we pass from the surface of the fluid. For $z=h$ this axis becomes $=2a$, and the vertical axis vanishes. That is, for particles of water at the bottom of the canal there is only a motion of translation backwards and forwards in lines of length$=2a$. For particles at the surface of the fluid and for $h=l$ the horizontal axis is nearly $=535.5a$, or the ratio between the lengths of the horizontal axis at top and bottom of the fluid is nearly 267.7.

Referring now to the values of u and w near the surface, we have

$$\left\{\begin{array}{l} u=-A\varepsilon^{-\frac{2\pi}{l}z}\cos.\frac{2\pi}{l}(\omega t+x) \\ w=A\varepsilon^{-\frac{2\pi}{l}z}\sin.\frac{2\pi}{l}(\omega t+x) \end{array}\right.$$

Differentiating these for t, squaring and adding the results, and we obtain the expression

$$\left(\frac{du}{dt}\right)^2+\left(\frac{dw}{dt}\right)^2=\left(\frac{2\pi\omega}{l}\mathrm{R}\right)^2$$

The quantity on the left hand side of this equation gives the square of the velocity of the fluid particle in its circular path; this, as we see from the second member of the equation, is independent of the time, and is directly proportional to the radius of the circle; but the radius depends upon the quantity $\varepsilon^{-\frac{2\pi}{l}z}$ for its value, and this increases as z decreases—therefore, for particles near the surface of shallow water, we have the velocity x varies inversely as their depth. From this it is evident that the water at the top of the wave moves most rapidly forward, while that at the bottom moves most rapidly backward. In the expression for ω discarding the terms $\varepsilon^{-\frac{2\pi}{l}h}$ we have for the velocity of translation of particles near the surface of water of finite depth, or anywhere within the mass of a body of water of infinite depth

$$\omega = \sqrt{\frac{gl}{2\pi}}.$$

Substituting for ω its value of $\frac{l}{\tau}$ and we find for τ the value

$$\sqrt{\frac{2\pi l}{g}}.$$

Calling t' the time of oscillation of a simple pendulum of length l, we have

$$\tau = \sqrt{\frac{2}{\pi}} t'.$$

From the above value of ω we see that the velocity of transmission of the wave varies as the square root of the length. In all cases, indeed, the velocity is nearly as the square root of the length, for the factor

$$\frac{\varepsilon^{\frac{2\pi}{l}h} - \varepsilon^{-\frac{2\pi}{l}h}}{\varepsilon^{\frac{2\pi}{l}h} + \varepsilon^{-\frac{2\pi}{l}h}}.$$

is nearly equal to unity. In the case of very shallow water, the velocity diminishes considerably—as the quantity just written decreases rapidly with h—vanishing as is obvious for $h=o$.

So far we have confined ourselves to a single wave, that is, to a single value of φ satisfying the equation $\Delta^2\varphi=o$. But we have seen that if there are several values of φ each satisfying this equation, that collectively they satisfy the equation

$$\Delta^2\Sigma\varphi=o.$$

In the case when the wave lengths are the same but the phases different, we can easily find the result of adding together the waves given by the functions φ, φ_1, . . . φ_i.

The value that we have already obtained for φ may be written

$$-\frac{a_0}{\sigma}\left\{\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right\}\sin.\sigma(\omega t+x+a_0)$$

where it is to be understood that $a_0=o$ and is merely introduced for future symmetry. Any other function φ_i which satisfies the equation $\Delta^2\varphi=o$, may be written under the above conditions

$$\varphi_i = -\frac{a_i}{\sigma}\left(\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right)\sin.\sigma\,(\omega t + x + a_i)$$

And it is not difficult to see that a summation of these functions will give us

$$\Sigma\varphi = -\frac{A}{\sigma}\left\{\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right\}\sin.\,\sigma\,(\omega t + x + \Psi)$$

When

$$A^2 = \sum_{i=o} a_i^2 + 2\sum_{j=o} a_j \sum_{k=j+1} a_k \cos.\sigma\,(a_{k-1} - a_k)$$

and

$$\tan.\Psi = \frac{\sum\limits_{i=1} a_i \sin.\ \sigma a_i}{\sum\limits_{j=o} a_i \cos.\ \sigma aj}$$

or inversely

$$\Psi = \frac{1}{\sigma}\tan.^{-1}\ \frac{\sum\limits_{i=1} a_i \sin.\ \sigma a_i}{\sum\limits_{j=0} a_j \cos.\ \sigma a_j}$$

If any of the quantities σa, &c.,$=\pi$ or $(2n+1)\pi$ a change takes place in the summation. Suppose $\sigma a_i = \pi$ then φ_i becomes $-\varphi_i$ and is subtracted instead of added to the other functions; but if $\sigma a_i = (2n+1)\pi$ we have since $\sigma = \frac{2\pi}{\iota}$

$$a_i = (2n+1)\frac{l}{2}.$$

That is if the difference of phase is an odd multiple of half the wave length, the corresponding wave function is to be subtracted instead of added to the others in finding the resultant of the system.

Suppose now that we have two waves of the same length and amplitude, but of different phases and moving in opposite directions; the wave functions are obviously

$$\varphi = -\frac{a}{\sigma}\left\{ \varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)} \right\} \sin.\ \sigma\ (\omega t + x)$$

$$\varphi_1 = -\frac{a}{\sigma}\left\{ \varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)} \right\} \sin.\ \sigma\ (\omega t - x + a)$$

adding we have

$$\varphi + \varphi_1 = -\frac{a}{\sigma}\left\{ {}^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)} \right\} \sin.\ \sigma\ (\omega t + x) + \sin.\ \sigma\ (\omega t - x + a)$$

expanding the trigonometric factor and reducing by aid of the relations

$$\cos.\ a = 2 \cos.^2 \frac{a}{2} - 1,\ \sin.\ a = 2 \sin \frac{a}{2} \cos. \frac{a}{2},$$

we readily find this expression to become

$$\varphi+\varphi_1=-\frac{2a}{\sigma}\left\{\varepsilon^{(\sigma h-z)}+\varepsilon^{-\sigma(h-z)}\right\}\sin.\ \sigma\left(wt+\frac{a}{2}\right)\cos.\ \sigma\left(x-\frac{a}{2}\right).$$

From this we obtain by differentiation the values of the displacements

$$u=2a\left\{\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right\}\sin.\ \sigma\left(x-\frac{a}{2}\right)\sin.\ \sigma\left(wt+\frac{a}{2}\right)$$

$$w=2a\left\{\varepsilon^{\sigma(h-z)}-\varepsilon^{-\sigma(h-z)}\right\}\sin.\ \sigma\left(wt+\frac{a}{2}\right)\cos.\ \sigma\left(x-\frac{a}{2}\right).$$

Dividing the first of these by the second we have

$$\frac{u}{w}=\frac{\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}}{\varepsilon^{\sigma(h-z)}-\varepsilon^{-\sigma(h-z)}}\tan.\sigma\left\{x-\frac{a}{2}\right\};$$

this ratio is independent of the time, consequently each particle moves in a straight line the inclination of which varies with x and z. Since also this ratio is the same at any given point, no matter what be the time, the wave is a standing wave or has no progressive motion.

Hence if there exist in the liquid two waves having the same length and amplitude but moving in opposite directions the result is a single standing wave in which the particles move constantly in right lines whose inclinations to the axes vary with x and z.

Reverting now to our values of u and w, suppose that $u=o$, that is, that there be no horizontal motion, this gives us

$$\cos. \frac{2\pi}{l}(\omega t+)=o$$

or $$\omega t+x=\frac{l}{4} \text{ or } \frac{3l}{4} \text{ or } \frac{(2n+1)l}{4}.$$

That is, there is no horizontal motion at the nodes of the wave. The greatest horizontal motion evidently corresponds to

$$\omega t+x=o, \frac{l}{2}, \text{ or } l$$

or the greatest horizontal motion is at the highest point of the crest and the lowest point of the trough of the wave—

and evidently the motions at these points are in opposite directions—which we have seen before from other considerations.

In like manner by making $w=o$ we find that at the top and bottom of the wave there is no vertical motion. Also, that the greatest vertical motion is at the nodes of the wave.

Similar results are obtained by examining the equations a. There is no horizontal motion at the points when $x-\frac{a}{2}=\frac{l}{4}$ or $\frac{3l}{4}$ but there is a maximum of vertical motion; also, there is no vertical motion at the points where $x-\frac{a}{2}=o, \frac{l}{2}$, or l, but there is the greatest horizontal motion.

Airy has shown in his treatise on "Tides and Waves," that if the channel is of variable depth or width, that waves of the nature just described, that is, waves caused by the simple oscillation of the particles, could not exist by themselves, but require for their existence the action of certain exterior forces into the nature of which it is not necessary here to go. Without going into a mathematical discussion of the reflection of waves, I will merely state that after impinging upon a wall the particles of the wave move up and down the surface through a distance equal to twice their previous vertical displacement, and the same with particles at a distance of half a wave length from the wall; particles at a distance from the wall of one quarter of a wave length, merely vibrate in a horizontal direction. When a series of waves enters shallow water the period remains the same, but the velocity and wave length diminish; the front of the wave becomes steeper than the back, and continues to become more and more abrupt until the top of the wave curls over the front and the wave breaks in surf on the beach.

§ 4.

CYLINDRICAL WAVES.

If we throw a pebble into a body of water, or if we simply bring a solid body in contact with the water at one point we know that a series of waves is generated which are circular in form, concentric and having their center at the point where the disturbance takes place. The waves thus generated are called cylindrical waves, and the line passing through the center of these circles and normal to the surface of the fluid is called the wave-axis, and evidently is the geometrical axis of the concentric cylinders.

In the case of such waves as this it is evidently not admissible to assume the displacement in any direction as equal to zero, there will clearly be motion in the direction of all these axes. Our axis of z will be assumed as having the same direction as in the foregoing section, and the axes of X and Y will lie in the surface of the fluid at rest. Our equation of continuity will have the general form

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}=o$$

the displacements being of course given by

$$u=\frac{d\varphi}{dx},\ v=\frac{d\varphi}{dy},\ w=\frac{d\varphi}{dz}.$$

The same remarks that were previously made concerning the form of φ will hold here, the waves being supposed to emanate from the wave axis, so we can write for φ the equation

$$\varphi=\varepsilon^{\pm\sigma z} f\ (x, y) \begin{matrix}\sin\\ \cos\end{matrix} \left\{\frac{2\pi}{\tau} t\right\}$$

when τ as before denotes the periodic time. If the wave axis be taken as the axis of z we have, r denoting the distance from this axis to any point in a plane parallel to the plane of $x\ y$,

$$r^2=x^2+y^2$$

and we may write φ in the form,

$$\varphi=\varepsilon^{\pm\sigma z} f\ (r) \begin{matrix}\sin\\ \cos\end{matrix} \left\{\frac{2\pi}{\tau} t\right\}$$

We must now, as before, determine the form of f. Substitute this value of φ in the equation of continuity, and it is easily found to reduce to the form

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}+\sigma^2 f=o$$

Transforming this by the substitution

$$r=\frac{s}{\sigma}$$

we obtain a known form

$$\frac{d^2f}{ds^2}+\frac{1}{s}\frac{df}{ds}+f=o.$$

This is a particular case of the more general equation,

$$\frac{d^2f}{ds^2}+\frac{1}{s}\frac{df}{ds}+\left(1-\frac{n^2}{s^2}\right)f=o$$

of which a particular solution is the Bessel's function $J_n(s)$ given by

$$J_n(s)=$$

$$\frac{s_n}{2^n n!}\left\{1-\frac{s^n}{2(2n+2)}+\frac{s^4}{2.4(2n+2)(2n+4)}\right.$$

$$\left.\frac{s^6}{2.4.6(2n+2)(2n+4)(2n+6)}+\ldots\right\}$$

For our case $n=o$, and the function $J_0(s)$ is a particular solution, viz.:

$$J_0(s)=\left\{1-\frac{s^2}{2^2}+\frac{s^4}{2^2.4^2}-\frac{s^6}{2^2.4^2.6^2}+\ldots\ldots\right\}$$

This is easily obtained directly, calling f_0 the particular solution sought assume

$$f_0=a+a_1s+a_2s^2+a_3s^3+a_4s^4+\ldots$$

Substituting in the differential equation, and we have

$$o=a_1s^{-1}+(a+4a_2)+(a_1+9a_3)s$$
$$+(a_2+16a_4)s^2+\dots.$$

from which we have

$$a_1=a_3=a_5=\dots\dots a_{2i+1}=o$$

$$a_2=-\frac{a}{4},\ a_4=\frac{a}{64}=\frac{a}{2^2.4^2},\ \&c.\dots.$$

when i is of course any positive integral. This gives us then for our particular solution

$$f_0=J_0(s)=a\left\{1-\frac{s^2}{2^2}-\frac{s^4}{2^2.4^2}+\frac{s^6}{2^2.4^2.6^2}+\dots\right\}$$

Designate now by $Y_0(s)$ the other particular solution of the differential equation, and for brevity write simply Y_0 and J_0 instead of $Y_0(s)$ and $J_0(s)$. Let now Σ denote a function of s, then it is well known that we can write

$$Y_0=J_0\Sigma$$

Substituting this in the differential equation and it becomes

$$\left(\frac{1}{s}+\frac{2}{J_0}\frac{dJ_0}{ds}\right)\frac{d\Sigma}{ds}+\frac{d^2\Sigma}{ds^2}=o$$

Dividing this by $\frac{d\Sigma}{ds}$ and integrating, gives

$$\log. s + 2 \log. J_0 + \log. \frac{d\Sigma}{ds} = \text{const.}$$

or assuming the const. $= o$, and passing to exponentials

$$sJ_0^2 \frac{d\Sigma}{ds} = 1$$

from which

$$\frac{d\Sigma}{ds} = \frac{1}{sJ_0^2}$$

and

$$\Sigma = \int \frac{ds}{sJ_0^2}$$

and, by substitution in the equation giving Y_0,

$$Y_0 = J_0 \int \frac{ds}{sJ_0^2}.$$

Now from the value of J_0 it is clear that the expansion of $\frac{1}{J_0^2}$ can only contain even powers of s and we can thus write

$$\frac{1}{sJ_0^2} = \frac{1}{s}\left\{1 + As^2 + Bs^4 + Cs^6 + \right\};$$

multiplying by ds and integrating gives

$$Y_0=J_0 \log. s+J_0\left\{A\frac{s^2}{2}+B\frac{s^4}{4}+C\frac{s^6}{6}+\right\}$$

or as it may be written for brevity

$$Y_0=J_0 \log. s+E_0$$

The quantity E_0 is the product of two infinite series each of which contains only positive integral powers of s and, consequently, according to a principle in the theory of the Bessel's functions can be developed in a series of these functions and, moreover, as all the powers in J_0 and the other factor of E_0 are even, only the even Bessel's functions will appear in the development thus

$$Y_0=J_0 \log. s+aJ^0+bJ^2+cJ^4+\ldots.$$

The co-efficients a, b, c have to be determined.

Take again the differential equation

$$\frac{d^2f}{ds^2}+\frac{1}{s}\frac{df}{ds}+f=o$$

and perform the operation

$$\frac{d^2}{ds^2}+\frac{1}{s}\frac{d}{ds}+1$$

on the quantity $J_0 \log. s$ and we find

$$\left\{\frac{d^2}{ds^2}+\frac{1}{s}\,\frac{d}{ds}+1\right\}\mathrm{J}_0 \log.\ s=\frac{2}{s}\,\frac{d\mathrm{J}_0}{ds}.$$

Represent the operator for brevity by Δ, then this is

$$\Delta \mathrm{J}_0 \log.\ s=\frac{2}{s}\,\frac{d\mathrm{J}^0}{ds}$$

We have now from the general differential equation affording Bessel's functions

$$\Delta \mathrm{J}_n=\frac{n^2}{s^2}\mathrm{J}_n$$

and for $n=o$

$$\Delta \mathrm{J}_o=o$$

and for $n>o$

$$\Delta \mathrm{J}_n=\frac{n}{2s}\left\{\mathrm{J}_{n-1}+\mathrm{J}_{n+1}\right\}$$

according to a known relation connecting these three consecutive functions. And so we have, finally

$$\Delta \mathrm{J}_o \log.\ s=-\frac{2}{s}\,\mathrm{J}_1$$

Now from the above value of Y_0 we have

$$\Delta\,\mathrm{Y}_0=\Delta(\mathrm{J}_0 \log.\ s)+a\Delta \mathrm{J}_0 +b\Delta \mathrm{J}_2+c\Delta \mathrm{J}_4+\dots.$$

And by aid of the transformations just given

$$\varDelta Y_0 = -\frac{2}{s} J_1 + \frac{b}{s}(J_1 + J_3) + \frac{2c}{s} \cdot (J_3 + J_5)$$
$$+ \frac{3d}{s}(J_5 + J_7) \times ..$$

Now Y_0 being a particular solution of the differential equation $\varDelta f = o$, we must have $\varDelta Y_0 = o$; this enables us to find

$b = -2c = 3d = -4c = 5g = \&c. \ldots\ldots\ldots = 2$

and by substitution

$$Y_0 = J_0 \log. s + aJ_0$$
$$+ 2[J_2 - \tfrac{1}{2}J_4 + \tfrac{1}{3}J_6 - \tfrac{1}{4}J_8 + \ldots.]$$

The complete solution of the differential equation $\varDelta f = o$, *i.e.*,

$$\frac{d^2 f}{ds^2} + \frac{1}{s}\frac{df}{ds} + f = o$$

is now given by

$$f = AJ_0 + BY_0$$

or

$$= (A + B \log. z)\, J_0 + BE_0$$

It maybe verified without much difficulty that the quantity E_0 is of the form

$$E_0 = \left\{ \frac{s^2}{2^2} - \frac{1 + \frac{1}{2}}{2^2.4^2} s_4 + \frac{1 + \frac{1}{2} + \frac{1}{3}}{2^2.4^2.6^2} s^6 + \ldots.. \right\}$$

The quantities J_0 and Y_0 expressed in the form of definite integrals are—vide Boole's Diff. Equas.

$$J_0=\frac{1}{\pi}\int_0^\pi \cos(s\sin\omega)\,d\omega$$

$$Y_0=\frac{1}{\pi}\int_0^\pi \cos(s\sin\omega)\log(4\,s\cos^2\omega)\,d\omega$$

and we have for f by substitution

$$f=\frac{1}{\pi}\int_0^\pi \cos(s\sin\omega)\,(A+B\log(4s\cos^2\omega)\,d\omega$$

or as this may be written

$$f=\int_0^\pi \cos(s\sin\omega)\,(C+D\log(s\cos^2\omega)d\omega$$

when

$$C=\frac{A+B\log.\ 4}{\pi},\ D=\frac{B}{\pi}.$$

Before going on to the application of these results to the problem in hand, we will investigate the change produced in the quantities J_0 and Y_0 by allowing s to become very great.

Instead of f in the differential equation $\Delta f=o$ write $f\sqrt{s}$ this equation thus becomes

$$\frac{d^2(f\sqrt{s})}{ds^2}+\left(1+\frac{1}{4s^2}\right)f\sqrt{s}=o$$

and this for s very large is simply

$$\frac{d^2(f\sqrt{s})}{ds^2}+f\sqrt{s}=o.$$

This equation gives on integration

$$f\sqrt{s}=a \cos s + b \sin s$$

or

$$f=\frac{a \cos s + b \sin s}{\sqrt{s}}$$

when a and b are of course constants. We have then obviously from this

$$J_o=\frac{\alpha \cos s+\beta \sin s}{\sqrt{s}}$$

$$Y_o=\frac{\alpha' \cos s+\beta' \sin s}{\sqrt{s}}$$

from which we can see that for infinitely great values of s the functions J_o and Y_o will vanish.

Now by substituting for s its value of σr we can, by taking as the argument of the functions thus obtained the quantity $-\frac{s^2}{2^2}$, or $-\frac{\sigma^2 r^2}{4}$ write f in the form

$$f = \left\{1+ \sum_{i=1}^{i=s} \frac{\theta^i}{(i!)^2}\right\}(A+B \log.2\sqrt{-\theta})$$

$$-B\sum_{i=1}^{i=x}\frac{\theta^i}{(i!)^2}.\sum_{j=1}^{j=i}\frac{1}{j}$$

For convenience of reference hereafter we shall write this in the form,

$$f=\mathrm{A}\,\mathrm{P}(\theta)+\mathrm{B}\,\Omega(\theta).$$

Substituting this value of f in the expression for φ we obtain,

$$\varphi=\varepsilon^{\pm\sigma z}[\mathrm{A}\mathrm{P}(\theta)+\mathrm{B}\Omega(\theta)]\left\{\sin\frac{2\pi}{\tau}t+\cos\frac{2\pi}{\tau}t\right\}$$

or expanding this and writing instead of A and B, the quantities a_1, b_1, a_2, b_2, α_1, β_1, α_2, β_2, we have

$$\varphi=\mathrm{P}\sin\frac{2\pi}{\tau}t\left(a_1\varepsilon^{\sigma z}+\alpha_1\varepsilon^{-\sigma z}\right)+\mathrm{P}\cos\frac{2\pi}{\tau}t\left(a_2\varepsilon^{\sigma z}+\alpha_2\varepsilon^{-\sigma z}\right)$$
$$+\Omega\sin\frac{2\pi}{\tau}t\left(b_1\varepsilon^{\sigma z}+\beta_1\varepsilon^{-\sigma z}\right)+\Omega\cos\frac{2\pi}{\tau}t\left(b_2\varepsilon^{\sigma z}+\beta_2\varepsilon^{-\sigma z}\right)$$

This gives us for u, v, w the following values;

$$u=\left\{\sin\frac{2\pi}{\tau}t\left(a_1\varepsilon^{\sigma z}+a_1\varepsilon^{-\sigma z}\right)+\cos\frac{2\pi}{\tau}t\left(a^2\varepsilon^{\sigma z}+a_2\varepsilon^{-\sigma z}\right)\right\}\frac{d\mathrm{P}}{dx}$$

$$+\left\{\sin\frac{2\pi}{\tau}t\left(b_2\varepsilon^{\sigma z}+\beta_2\varepsilon^{-\sigma z}\right)\right.$$

$$\left.+\cos\frac{2\pi}{\tau}t\left(b_2\varepsilon^{\sigma z}+\beta_2\varepsilon^{-\sigma z}\right)\right\}\frac{d\Omega}{dx},$$

$$v=\left\{\sin\frac{2\pi}{\tau}t\left(a_1\varepsilon^{\sigma z}+\alpha_1\varepsilon^{-\sigma z}\right)\right.$$

$$\left.+\cos\frac{2\pi}{\tau}t\left(a_2\varepsilon^{\sigma z}+\alpha_2\varepsilon^{-\sigma z}\right)\right\}\frac{dP}{dy}$$

$$+\left\{\sin\frac{2\pi}{\tau}t\left(b_1\varepsilon^{\sigma z}+\beta_1\varepsilon^{-\sigma z}\right)\right.$$

$$\left.+\cos\frac{2\pi}{\tau}t\left(b_2\varepsilon^{\sigma z}+\beta_2\varepsilon^{-\sigma z}\right)\right\}\frac{d\Omega}{dy},$$

$$w=\mathrm{P}\sin\frac{2\pi}{\tau}t\,\sigma\left(a_1\varepsilon^{\sigma z}-\alpha_1\varepsilon^{-\sigma z}\right)$$

$$+\mathrm{P}\cos\frac{2\pi}{\tau}t\,\sigma\left(a_2\varepsilon^{\sigma z}-\alpha_2\varepsilon^{-\sigma z}\right)$$

$$+\Omega\sin\frac{2\pi}{\tau}t\,\sigma\left(b_1\varepsilon^{\sigma z}-\beta_1\varepsilon^{-\sigma z}\right)$$

$$+\Omega\cos\frac{2\pi}{\tau}t\,\sigma\left(b_2\varepsilon^{\sigma z}-\beta_2\varepsilon^{-\sigma z}\right)$$

The expression for the fluid pressure is obtained here in the same manner as in the case of plane waves, and is

$$p=\rho\left\{\left(\frac{2\pi}{\tau}\right)^2\varphi+gz\right\}+\text{const.}$$

Writing as before $z=z_0+w$, remembering that σ and w are quantities of the first order of magnitude, and so discarding terms containing σw or higher orders, we have

$$p=p_0+\rho y z_0+\rho\left\{\pi\sin\frac{2\pi}{\tau}t\right.$$

$$\left\{\left(\frac{2\pi}{\tau}\right)^2 a_1\varepsilon^{\sigma z_0}+\alpha\varepsilon^{-\sigma z_0}\Big)+y\sigma\left(a_1\varepsilon^{\sigma z_0}-\alpha_1\varepsilon^{-\sigma z_0}\right)\right\}$$

$$+\mathrm{P}\cos\frac{2\pi}{\tau}t$$

$$\left\{\left(\frac{2\pi}{\tau}\right)^2\left(a_2\varepsilon^{\sigma z_0}+\alpha\varepsilon^{-\sigma z_0}\right)+y\sigma\left(a_2\varepsilon^{\sigma z}-\alpha_2\varepsilon^{-\sigma z_0}\right)\right\}$$

$$+\Omega\sin\frac{2\pi}{\tau}\tau$$

$$\left\{\left(\frac{2\pi}{\tau}\right)^2\left(b_1\varepsilon^{\sigma z_0}+\beta_1\varepsilon^{-\sigma z_0}\right)+y\sigma\left(b_1\varepsilon^{\sigma z_0}-\beta_1\varepsilon^{-\sigma z_0}\right)\right\}$$

$$+\Omega\cos\frac{2\pi}{\tau}t$$

$$\left\{\left(\frac{2\pi}{\tau}\right)^2\left(b_1\varepsilon^{\sigma z_0}+\beta_2\varepsilon^{\sigma-z_0}\right)+y\sigma\left(b_2\varepsilon^{\sigma z_0}-\beta_2\varepsilon^{\sigma-z_0}\right)\right\}$$

For particles on the surface of the fluid at rest we have, of course, $z_0=o$ and $p=p_0$. This gives us

$$\mathrm{P}\sin.\frac{2\pi}{\tau}t\left\{\left(\frac{2\pi}{\tau}\right)^2(a_1+\alpha_1)+g\sigma(a_1-\alpha_1)\right\}$$

$$+\mathrm{P}\cos\frac{2\pi}{\tau}t\left\{\left(\frac{2\pi}{\tau}\right)^2(a_2+\alpha_2)+g\sigma(a_2-\alpha_2)\right\}$$

$$+\Omega\sin\frac{2\pi}{\tau}t\left\{\left(\frac{2\pi}{\tau}\right)^2[b_1+\beta_1]\right.$$

$$\left.+g\sigma\Big(b_1-\beta_1\Big)\right\}$$

$$+\Omega\cos\frac{2\pi}{\tau}t\left\{\left(\frac{2\pi}{\tau}\right)^2\left\{b_2+\beta_2\right\}\right.$$

$$\left.+g\sigma\Big(b_2-\beta_2\Big)\right\}=o$$

In order that this may be satisfied we must have obviously

$$\frac{a_1-\alpha_1}{a_1+\alpha_1}=\frac{a_2-\alpha_2}{a_2+\alpha_2}=\frac{b_1-\beta_1}{b_1+\beta_1}=\frac{b_2-\beta_2}{b_2+\beta_2}=\frac{\left(\frac{2\pi}{\tau}\right)^2}{g\sigma}$$

from which we obtain,

$$\frac{a_1}{\alpha_1}=\frac{a_2}{\alpha_2}=\frac{b_1}{\beta_1}=\frac{b_2}{\beta_2}$$

We can now write

$$w=\sigma\mathrm{P}\sin\frac{2\pi}{\tau}t\left(a_1\varepsilon^{\sigma z}-a_1\varepsilon^{-\sigma z}\right)$$

$$+c_1\sigma\mathrm{P}\cos\frac{2\pi}{\tau}t\left(a_1\varepsilon^{\sigma z}-a_1\varepsilon^{-\sigma z}\right)$$

$$+c_2\sigma\Omega\sin\frac{2\pi}{\tau}t\left(a_1\varepsilon^{\sigma z}-a_1\varepsilon^{-\sigma z}\right)$$

$$+c_3\sigma\Omega\cos\frac{2\pi}{\tau}t\left(a_1\varepsilon^{+\sigma z}-a_1\varepsilon^{-\sigma z}\right)$$

when the meaning of the constants c_1, c_2, c_3 is obvious. Introducing now the condition that $w=o$ for $h=o$ we have as for plane waves

$$\frac{a_1}{a_1}=\frac{\varepsilon^{-\sigma h}}{\varepsilon^{\sigma h}};$$

This gives us again

$$\left(\frac{2\pi}{\tau}\right)^2=g\sigma\frac{\varepsilon^{\sigma h}-\varepsilon^{-\sigma h}}{\varepsilon^{\sigma h}+\varepsilon^{-\sigma h}}.$$

We have now for φ the equation,

$$\begin{aligned}\varphi=a_1\mathrm{P}\sin\frac{2\pi}{\tau}t\left(\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right)\\ +b_1\mathrm{P}\cos\frac{2\pi}{\tau}t\left(\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right)\\ +a_2\Omega\sin\frac{2\pi}{\tau}t\left(\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right)\\ +b_2\Omega\cos\frac{2\pi}{\tau}t\left(\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right)\end{aligned}$$

when a_1, b_1, a_2, b_2, are new constants, whose values are respectively,

$$\frac{a_1}{\varepsilon^{\sigma h}},\ \frac{c_1a_1}{\varepsilon^{\sigma h}},\ \frac{c_2a_1}{\varepsilon^{\sigma h}},\ \frac{c_3a_1}{\varepsilon^{\sigma h}},$$

or those multiplied by any arbitrary constant. This value of φ may be written in the form

$$\varphi = \left(\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right)$$

$$\left[(a_1 P + a_2 \Omega)\sin\frac{2\pi}{\tau}t + (b_1 P + b_2 \Omega)\cos\frac{2\pi}{\tau}t\right]$$

from which we obtain

$$u = \left(\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right)\left\{\left(a_1\frac{dP}{dx} + a_2\frac{d\Omega}{dx}\right)\right.$$

$$\left.\sin\frac{2\pi}{\tau}t + \left(b_1\frac{dP}{dx} + b_2\frac{d\Omega}{dx}\right)\cos\frac{2\pi}{\tau}t\right\}$$

$$v = \left(\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right)\left\{\left(a_2\frac{dP}{dy} + a_2\frac{d\Omega}{dy}\right)\right.$$

$$\left.\sin\frac{2\pi}{\tau}t + \left(b_1\frac{dP}{dy} + b_2\frac{d\Omega}{dy}\right)\cos\frac{2\pi}{\tau}t\right\}$$

$$w = -\sigma\left(\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right)$$

$$\left[(a_1 P + a_2 \Omega)\sin\frac{2\pi}{\tau}t + (b_1 P + b_2 \Omega)\cos\frac{2\pi}{\tau}t\right]$$

We have, however,

$$\frac{dP}{d} = \frac{dP}{d\theta}\cdot\frac{d\theta}{dx} = \frac{dP}{d\theta}\cdot\frac{d}{dx}\left(-\frac{\sigma^2 r^2}{4}\right)$$

$$= -\frac{\sigma^2 x}{2}\,\frac{dP}{d\theta}, \text{ \&c.} \ldots\ldots$$

consequently,

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$$u=-\frac{\sigma^2 x}{2}\left(\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right)\left\{\left(a_1\frac{dP}{d\theta}+a_2\frac{d\Omega}{d\theta}\right)\sin\frac{2\pi}{\tau}t+\left(b_1\frac{dP}{d\theta}+b_2\frac{d\Omega}{d\theta}\right)\cos\frac{2\pi}{\tau}t\right\}$$

$$v=-\frac{\sigma^2 y}{2}\left(\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right)\left\{\left(a_1\frac{dP}{d\theta}+a_2\frac{d\Omega}{d\theta}\right)\sin\frac{2\pi}{\tau}t+\left(b_1\frac{dP}{d\theta}+b_2\frac{d\Omega}{d\theta}\right)\cos\frac{2\pi}{\tau}t\right\}$$

From these equations we see that the path of the particle is always in a plane passing through itself and the axis of z.

The expression for p becomes now

$$p=p_0+\rho g z_0+2\rho g\sigma\left\{\left(a_1\pi+a_2\Omega\right)\sin\frac{2\pi}{\tau}t+(b_1\pi+b_2\Omega)\cos\frac{2\pi}{\tau}t\right\}\frac{\varepsilon^{\sigma z_0}-\varepsilon^{-\sigma z_0}}{\varepsilon^{\sigma h}+\varepsilon^{-\sigma h}}$$

The values which have been obtained for the displacements and the fluid pressure afford the complete solution of the problem under consideration.

The results obtained are, however, very much modified in the cases where the particles are removed to great distances from the axis. We have already seen the change produced in the function f in such a case, viz., this quantity becomes

$$f=\frac{a\cos s+b\sin s}{\sqrt{s}}$$

or since σ is a constant

$$f=\frac{A\cos\sigma r+B\sin\sigma r}{\sqrt{r}}.$$

We might have so transformed our first obtained value of f that the infinite series therein contained should have proceeded according to ascending powers of $\frac{1}{r^2}$ and thus obtained the same result; this, however, would have been a difficult process.

The quantities P, and Ω are now given by the equations,

$$P=\frac{\sin\sigma r}{\sqrt{r}},\ \Omega=\frac{\cos\sigma r}{\sqrt{r}}$$

Substituting these in our value for φ and we have,

$$\varphi=\frac{1}{\sqrt{r}}\left\{\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right\}$$
$$\left\{(a_1\sin\sigma r+a_2\cos\sigma r)\sin\frac{2\pi}{\tau}t\right.$$
$$\left.+(b_1\sin\sigma r+b_2\cos\sigma r)\cos\frac{2\pi}{\tau}t\right\}$$

for the simplest case of waves we can make $a_1 = b_2 = o$ and $a_2 = b_1 = a$; thus

$$\varphi = \frac{a}{\sqrt{r}} \left\{ \varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)} \right\} \sin\left(\frac{2\pi}{\tau} t + \sigma r\right)$$

Differentiating this for r gives us the radial velocity of a particle, representing this by η and we have

$$\eta = \frac{1}{\sqrt{r}} \left(\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)} \right)$$

$$\left\{ \sigma a \cos\left(\frac{2\pi}{\tau} t + \sigma r\right) + \frac{a}{2r} \sin\left(\frac{2\pi}{\tau} t + \sigma r\right) \right\}$$

discarding term containing $\frac{1}{r}$,

$$\eta = \frac{\sigma a}{\sqrt{r}} \left(\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)} \right) \cos\left(\frac{2\pi}{\tau} t + \sigma r\right)$$

also

$$w = -\frac{\sigma a}{\sqrt{r}} \left(\varepsilon^{\sigma(h-z)} - \varepsilon^{-\sigma(h-z)} \right) \sin\left(\frac{2\pi}{\tau} t + \sigma r\right)$$

The expression for p also becomes, after making the proper substitutions and reductions,

$$p = p_0 + \rho g z_0 + 2\rho g \frac{\sigma a}{\sqrt{r}} \frac{\varepsilon^{\sigma z_0} - \varepsilon^{-\sigma z_0}}{\varepsilon^{\sigma h} + \varepsilon^{-\sigma h}},$$

$$\sin\left(\frac{2\pi}{\tau} t + \sigma r\right).$$

Introducing now the wave length l we have for great values of r

$$\frac{1}{\sqrt{r+l}}=\frac{1}{\sqrt{r}}$$

and also for a first approximation,

$$\frac{1}{\sqrt{r}}=\frac{1}{\sqrt{r_0+\eta}}=\frac{1}{\sqrt{r_0}}$$

we have, as before, $\sigma=\frac{2\pi}{l}$, and $l = \tau\ \omega$. Thus φ becomes now,

$$\varphi=\frac{a}{\sqrt{r_0}}\left(\varepsilon^{\frac{2\pi}{l}(h-z)}+\varepsilon^{-\frac{2\pi}{l}(h-z)}\right)\sin\frac{2\pi}{l}(\omega t+r)$$

with as before,

$$\omega=\sqrt{\frac{lg}{2\pi}\,\frac{\varepsilon^{\sigma h}-\varepsilon^{-\sigma h}}{\varepsilon^{\sigma h}+\varepsilon^{-\sigma h}}}$$

which for great depths or for particles near the surface becomes,

$$\sqrt{\frac{lg}{2\pi}}.$$

The same deductions are to be made here as in the case of plane waves, viz., that for particles any where within the mass of a fluid of infinite depth or near the surface of a mass of finite depth the velocity varies as the square root of the wave length. Write now $a\sigma = -a$ and collect all of our expressions:

$$\varphi = -\frac{l}{2\pi}\frac{a}{\sqrt{r_0}}\left\{\varepsilon^{\frac{2\pi}{l}(h-z)} + \varepsilon^{-\frac{2\pi}{l}(h-z)}\right\} \sin\frac{2\pi}{l}\left(\omega t + r\right)$$

$$\eta = -\frac{a}{\sqrt{r_0}}\left\{\varepsilon^{\frac{2\pi}{l}(h-z)} + \varepsilon^{-\frac{2\pi}{l}(h-z)}\right\} \cos\frac{2\pi}{l}\left(\omega t + r\right)$$

$$w = \frac{a}{\sqrt{r_0}}\left\{\varepsilon^{\frac{2\pi}{l}(h-z)} - \varepsilon^{-\sigma(h-z)}\right\} \sin\frac{2\pi}{l}\left(\omega t + r\right)$$

$$p = p_0 + \rho g z_0 - 2\rho g\frac{a}{\sqrt{r_0}}\frac{\varepsilon^{\frac{2\pi}{l}z_0} - \varepsilon^{-\frac{2\pi}{l}z_0}}{\varepsilon^{\frac{2\pi}{l}h} + \varepsilon^{-\frac{2\pi}{l}h}} \sin\frac{2\pi}{l}\left(\omega t + r\right)$$

$$\omega=\sqrt{\frac{lg}{2\pi}\cdot\frac{\varepsilon^{\frac{2\pi}{l}h}-\varepsilon^{-\frac{2\pi}{l}h}}{\varepsilon^{\frac{2\pi}{l}h}+\varepsilon^{-\frac{2\pi}{l}h}}}$$

We see from these expressions that the amplitudes of cylindrical waves differ only from those of plane waves by the factor $\frac{1}{\sqrt{r_0}}$—or in cylindrical waves the amplitudes varies inversely as $\sqrt{r}$—and for particles very remote from the axis the amplitudes will vanish; whereas in the case of plane waves we saw that the amplitudes were always the same for the same depth.

From the expressions for the displacements we have as before

$$\frac{\eta^2}{\frac{a_2}{r_0}\left\{\varepsilon^{\frac{2\pi}{l}(h-z)}+\varepsilon^{\frac{2\pi}{l}(h-z)}\right\}^2}+\frac{w^2}{\frac{a^2}{r_0}\left\{\varepsilon^{\frac{2\pi}{l}(h-z)}-\varepsilon^{\frac{2\pi}{l}(h-z)}\right\}^2}=1$$

Now if h be finite we have as before for particles within the mass of the fluid—except near the surface—that they move in ellipses whose plane is vertical and passing through the wave axis, and whose transverse axis is in the direction of r. Also the axes of the ellipse decrease as r_0 increases and for particles infinitely remote from the wave axis they vanish, or these particles are at rest. The axes also, as in the case of plane waves, continuously decrease as z increases, and for $z=h$ the transverse axis becomes $\frac{2a}{\sqrt{r^0}}$ and the vertical axis vanishes as it should do.

If z be very small as compared with h, the equation of our ellipse becomes

$$\frac{\eta^2}{A^2 \varepsilon^{-\frac{4\pi}{l}z}} + \frac{w^2}{A^2 \varepsilon^{-\frac{4\pi}{l}z}} = 1$$

when

$$A = \frac{a}{\sqrt{r_0}} \varepsilon^{\frac{2\pi}{l}h}.$$

This is the equation of a circle whose radius is

$$A\,\varepsilon^{-\frac{2\pi}{l}2}$$

That is, for particles near the surface of a mass of fluid of finite depth, or for particles any where within the mass of a fluid of infinite depth, the motion is in a circle. It is shown as in the case of plane waves that this circular motion is uniform. In fact, all the results that we have obtained for plane waves are transferable into the corresponding results for cylindrical waves by merely multiplying by the factor $\frac{1}{\sqrt{r_0}}$.

Suppose now that we have a series of n waves of the same wave length and amplitude but of different phases, starting from the same axis; let these waves be defined in the same manner as in the case of plane waves and we shall have for the resultant wave function

$$\Sigma\varphi = -\frac{A}{\sigma\sqrt{r_0}}\left\{\varepsilon^{\sigma(h-z)} + \varepsilon^{\sigma(h-z)}\right\} \sin\sigma(\omega t + r + \psi)$$

when,

$$A^2 = \sum_{i=o}^{i=n} a_1^{\,2} + 2\sum_{j=o}^{j=n} a_j \sum_{=j+1}^{k=n} a_k \cos\sigma\left(a_{k-1} - a_k\right)$$

and

$$\Psi = \frac{1}{\sigma} \tan^{-1} \frac{\sum\limits_{i=1}^{i=n} a_i \sin \sigma a_i}{\sum\limits_{j=o}^{j=n} a_j \cos \sigma a_j}$$

If the wave lengths are the same, but the amplitudes different by reason of different initial values of r_0, the change in the form of these quantities is very slight; they become

$$A^2 = \sum_{i=o}^{i=n} \frac{a_i^2}{\sqrt{r_i}} + 2 \sum_{j=o}^{j=n} \frac{a_j}{\sqrt{r_j}} \sum_{k=j+1}^{k=n} \frac{a_k}{\sqrt{r_k}} \cos \sigma \left(a_{k-1} - a_k\right)$$

and

$$\Psi = \frac{1}{\sigma} \tan^{-1} \frac{\sum\limits_{i=1}^{i=n} \frac{a_i}{\sqrt{r_i}} \sin \sigma a_i}{\sum\limits^{i=n} \frac{a_i}{\sqrt{r_j}} \cos \sigma a_j}$$

Suppose now that we have two waves of the same wave lengths and amplitudes, but of different phases and going in opposite directions. The resultant wave function will be

$$\varphi+\varphi'=-\frac{2a}{\sigma\sqrt{r_0}}\left\{\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right\}\sin\sigma\left(\omega t+\frac{a}{2}\right)\cos\sigma\left(r-\frac{a}{2}\right)$$

which corresponds to a standing wave. Differentiating for r and z we have for the displacement of η and w,

$$\eta=\frac{2a}{\sqrt{r_0}}\left\{\varepsilon^{\sigma(h-z)}+\varepsilon^{-\sigma(h-z)}\right\}\sin\sigma\left(\omega t+\frac{a}{2}\right)\sin\sigma\left(r-\frac{a}{2}\right)$$

$$w=\frac{2a}{\sqrt{r_0}}\left(\varepsilon^{\sigma(h-z)}-\varepsilon^{-\sigma(h-z)}\right)\sin\sigma\left(\omega t+\frac{a}{2}\right)\cos\sigma\left(r-\frac{a}{2}\right)$$

The ratio $\frac{\eta}{w}$ is independent of t, and we make the same deduction as before, that the particles move in right lines whose inclinations to the axes vary with z and r.

Suppose that we have a series of parallel wave axes, and that waves proceed from them having the same length and equal amplitudes but different phases. Let the wave functions be given as

$$\varphi_0 = -\frac{a_0}{\sigma\sqrt{r_0^{(0)}}}\left(\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right) \sin \sigma(\omega t + r^{(0)} + a_0)$$

$$\varphi_1 = -\frac{a_1}{\sigma\sqrt{r_0^{(1)}}}\left(\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right) \sin \sigma(\omega t + r^{(1)} + a_1$$

$$\vdots$$

$$\varphi_n = -\frac{a_n}{\sigma\sqrt{r_0^{(n)}}}\left\{\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right\} \sin \sigma(\omega t + r^{(1)} + a_n)$$

adding these we have

$$\Sigma\varphi = -\frac{1}{\sigma}\sum_{i=o}^{i=n} a_i \left\{\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right\} \left\{\frac{\sin \sigma(\omega t + r^{0} + a_0)}{\sqrt{r_0^{(0)}}} + \frac{\sin \sigma(\omega t + r^{(1)} + a_1)}{\sqrt{r_0^{(1)}}} \ldots\ldots + \frac{\sin \sigma(\omega t + r^{(n)} + a_n)}{\sqrt{r_0^{(n)}}}\right\}$$

from which

$$w = \sum_{i=o}^{i=n} a_i \left\{\varepsilon^{\sigma(h-z)} - \varepsilon^{-\sigma(h-z)}\right\} \left\{\frac{\sin \sigma(\omega t + r^{(0)} + a_1)}{\sqrt{r_0^{(0)}}} \ldots\ldots + \frac{\sin \sigma(\omega t + r^{(n)} + a_n)}{\sqrt{r_0^{(n)}}}\right\}$$

The waves may evidently so move that at certain points the vertical displacements shall be equal to zero. We can determine these points by placing the trigonometric factor of w equal to zero; thus

$$\frac{\sin\sigma\,(\omega t+r^{(o)}+a_0)}{\sqrt{r_0^{(o)}}}+\frac{\sin\sigma\,(\omega t+r^{(1)}+a_1)}{\sqrt{r_0^{(1)}}}+\dots\frac{\sin\sigma\,(\omega t+r^{(n)}+a_n)}{\sqrt{r_0^{(n}}}=o$$

This expression can be divided into two parts, one of which shall have for a factor $\sin\sigma\,\omega\,t$, and the other $\cos\sigma\,\omega\,t$.

$$\sin\sigma\,\omega\,t\left\{\frac{\cos\sigma\,(r^{(0)}+a_0)}{\sqrt{r_0^{(0)}}}+\frac{\cos\sigma\,(r^{(1)}+a_1)}{\sqrt{r_0^{(1)}}}+\dots\frac{\cos\sigma\,(r^{(n)}+a_n)}{\sqrt{r_0^{(n)}}}\right\}$$

$$+\cos\sigma\,\omega\,t\left\{\frac{\sin\sigma\,(r^{(o)}+a_0)}{\sqrt{r_0^{(o)}}}+\frac{\sin\sigma\,(r^{(1)}+a_1}{\sqrt{r_0^{(1)}}}+\dots\frac{\sin\sigma\,(r^{(n)}+a_n)}{\sqrt{r^{n}_{(o)}}}\right\}=o$$

Equate separately to zero the factors

multiplying sin $\sigma\ \omega\ t$ and cos $\sigma\ \omega\ t$, square and add the resulting equations and we have after some easy reductions,

$$\sum_{i=o}^{i=n}\frac{1}{r_i^{(i)}}+\sum_{j=o}^{j=n}\sum_{k=1}^{k=n}\cos\sigma\left\{\frac{(r^{(j)}-a_j)-(r_k-a_k)}{\sqrt{r_0^j\, r_0^k}}\right\}=o$$

For the simple case of $n=2$ or two wave axes we have since $a_0=o$

$$\frac{1}{r_0^{(o)}}+\frac{1}{r_0^{(1)}}+\frac{2}{\sqrt{r_0^{(o)}\, r_0^{(1)}}}\cos\sigma\,(r^{(o)}-r^{(1)}-a)=o$$

If $r_0^{(o)}=r_0^{(1)}$ this becomes

$$\frac{2}{r_0}+\frac{2}{r_0}\cos\sigma\,(-a_1)=o$$

If now $-\sigma a_1=\pm(2n+1)\pi$ this equation will be satisfied, *i.e.*, if

$$a=\mp(2n+1)\frac{l}{2}$$

Therefore if the difference of phase is an odd multiple of half the wave length the vertical displacement is zero—but only for the points for which $r_0^{(0)}=r_0^{(1)}$. The points defined by the equation

$$r_0^{(0)}=r_0^{(1)}$$

lie on a plane which from its relation to the waves may be called the plane of symmetry. We will now examine a little more closely the conditions at this plane of symmetry. We have

$$\varphi_0 = -\frac{a_0}{\sigma\sqrt{r_0}}\left\{\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right\} \sin\sigma(\omega t + r)$$

$$\varphi' = -\frac{a_1}{\sigma\sqrt{r_0^{(1)}}}\left\{\varepsilon^{\sigma(h-z)} + \varepsilon^{-\sigma(h-z)}\right\} \sin\sigma(\omega t + r' + a_1)$$

writing, for convenience, r' for $r^{(1)}$. We have also

$$u = \frac{d\varphi}{dr}\frac{dr}{dx} + \frac{d\varphi'}{dr}\frac{dr'}{dx}$$

$$v = \frac{d\varphi}{dr}\frac{dr}{dy} + \frac{d\varphi'}{dx'}\frac{dr'}{dy}$$

when $r^2 = x^2 + y^2$ and $r'^2 = (x-2a)^2 + y^2$. Now since

$$\sigma a_1 = \pm(2n+1)\pi$$

we will have for $r = r'$

$$\frac{d\varphi}{dr} = -\frac{d\varphi'}{dr'}$$

At the plane of symmetry $n = a$ so that

$$\frac{dr}{dx} = -\left(\frac{dr'}{dx}\right), \text{ and } \frac{dr}{dy} = \frac{dr'}{dy}\,;$$

Therefore for $r = r'$ we have

$$u=2\frac{d\varphi}{dx}\frac{dr}{dx},\quad v=o,\quad w=o.$$

That is, at the plane of symmetry the displacement perpendicular to it is twice as great as that due to either wave acting separately, and the displacements parallel to this plane and the vertical displacements are equal to zero. Suppose now, further, that $a_1=o$: then we have for $r=r'$,

$$\frac{d\varphi}{dr}=\frac{d\varphi}{dr'};$$

therefore,

$$u=o,\quad v=2\frac{d\varphi}{dr}\frac{dr}{dy},\quad w=2\frac{d\varphi}{dz},$$

From which we have—if there is no difference of phase between the waves from the parallel axes—at the plane of symmetry there is no displacement in the direction of the axis of X, *i. e.* in the direction perpendicular to this plane; also that the displacement parallel to the plane and the vertical displacement are twice as great as they would be if there was but one wave.

The reader who is interested in the

subject of wave motion will do well to read an article on the subject by Lord Rayleigh in the April number of the *Philosophical Magazine* for 1876. An article in the September number of the same publication for 1878, though not bearing directly upon the subject, will also be found to contain much that is of value and interest; the article referred to is entitled "Hydrodynamic Problems in reference to the Theory of Ocean Currents," by M. Zöppritz. The mathematical theory of wave motion remains pretty much as Airy left it when he completed his work on the subject—so no better reference can be given than to that work—for any one wishing to acquire a thorough knowledge of the subject.

§ 5.

FREE VORTEX MOTION.

We have already seen under what circumstances it is impossible for rotational motion to exist in a fluid mass. If the fluid in its initial condition has irrotational motion—or, if it be at rest, and motion is induced by a system of conservative forces, then the motion will always be irrotational; *i. e.*, if the quantity

$$udx + vdy + wdz$$

is at any time an exact differential it will always be one. The conditions for this quantity being an exact differential are

$$\frac{dw}{dy} - \frac{dv}{dz} = o, \quad \frac{du}{dz} - \frac{dw}{dx} = o, \quad \&c.$$

Suppose that these quantities are not equal to zero, but that we have

$$\xi = \frac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right),$$

$$\eta = \frac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right),$$

$$\zeta = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right);$$

The quantities ξ, η and ζ, as is well known, denote the components of angular velocity around the axes of x, y and z, respectively, of a particle whose velocities parallel to these axes are u, v, w. That these quantities should have certain definite values different from zero is, of course, the condition that vortex, or rotational motion exist in the liquid. These values of ξ, η and ζ, pre-suppose that we know the values of u, v and w. A problem that now immediately presents itself for solution is to find the values of u, v and w, supposing ξ, η and ζ to be given.

Assume three functions U, V and W such that

$$u=\frac{dW}{dy}-\frac{dV}{dz},$$

$$v=\frac{dU}{dz}-\frac{dW}{dx},$$

$$w=\frac{dV}{dx}-\frac{dU}{dy},$$

The quantities u, v and w must satisfy the equation

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=o.$$

It is found without difficulty that this equation will only be satisfied by the above values of u, v and w if the following conditions hold,

$$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}=-2\xi,$$

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=-2\eta,$$

$$\frac{d^2W}{dx^2}+\frac{d^2W}{dy^2}+\frac{d^2W}{dz^2}=-2\zeta,$$

$$\frac{dU}{dx}+\frac{dV}{dy}+\frac{dW}{dz}=o.$$

The integrals of the first three of these equations are well known to be given by

$$U=\frac{1}{2\pi}\int\int\int\frac{\xi'dx'dy'dz'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}},$$

$$V=\frac{1}{2\pi}\int\int\int\frac{\eta'dx'dy'dz'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}},$$

$$W=\frac{1}{2\pi}\int\int\int\frac{\zeta'dx'dy'dz'}{\sqrt{(x'-x)^2+(y'-y)^2+(z-z)^2}};$$

where $x',{}'y,z'$ are the co-ordinates of any other point in the vortex element, and $\xi,'\ \eta,'\ \zeta'$ are the angular velocities at this point; the denominator,

$$r=\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$$

denotes the distance between this point and the assumed point to which the U,V, W refer. Before going further it will be convenient to give two of Helmholtz's definitions. The line passing through any point and coinciding at all times in direction with the instantaneous axis of rotation of that point is called a *vortex line.* If we consider a number of vortex lines passing through every point in the perimeter of an infinitely small surface, they will cut from the rest of the fluid a filament which is called a *vortex filament,* or a vortex filament is an infinitely small filament of the fluid whose bounding surface is made up of vortex lines.

Now, in our equations giving U, V, W, the points x, y, z and x' y', z' are supposed to lie on the same vortex filament; we can represent an element of this filament by $d\tau$, then our equations become,

$$U=\frac{1}{2\pi}\int\frac{\xi' d\tau}{r},$$

$$V=\frac{1}{2\pi}\int\frac{\eta' d\tau}{r},$$

$$W=\frac{1}{2\pi}\int\frac{\zeta' d\tau}{r}.$$

where the integrations are of course extended over all the space which is supposed to be filled with vortex filaments. Now to examine the condition

$$\frac{dU}{dx}+\frac{dV}{dy}+\frac{dW}{dz}=o.$$

By differentiation we have

$$\frac{dU}{dx}=-\frac{1}{2\pi}\int\int\int\frac{(x-x')\,\xi'\,dx'dy'dz'}{r^3}$$

$$\frac{dV}{dy}=-\frac{1}{2\pi}\int\int\int\frac{(y-y')\,\eta'\,dx'dy'dz'}{r^3}$$

$$\frac{dW}{dz}=-\frac{1}{2\pi}\int\int\int\frac{(z-z')\,\zeta'\,dx'dy'dz'}{r^3}$$

Integrating by parts we have,

$$\frac{dU}{dx}=-\frac{1}{2\pi}\int\int\frac{\xi'dy'dz'}{r}$$
$$+\frac{1}{2\pi}\int\int\int\frac{1}{r}\frac{d\xi'}{dx'},dx'dy'dz'$$

$$\frac{dV}{dy}=-\frac{1}{2\pi}\int\int\frac{\eta'dx'dz'}{r}$$
$$+\frac{1}{2\pi}\int\int\int\frac{1}{r},\frac{d\eta'}{dy'},dx'dy'dz'$$

$$\frac{dW}{dz}=-\frac{1}{2\pi}\int\int\frac{\zeta'dx'dy'}{r}$$
$$+\frac{1}{2\pi}\int\int\int\frac{1}{r}\frac{d\xi'}{dz'}\,dx'dy'dz'$$

Now since

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0$$

throughout the entire mass of the fluid we have

$$\frac{dU}{dx}+\frac{dV}{dy}+\frac{dW}{dz}=-\frac{1}{2\pi}\left\{\int\int\frac{\xi' dy' dz'}{r}\right.$$
$$\left.+\int\int\frac{\eta' dx' dz'}{r}+\int\int\frac{\zeta' dx' dy'}{r}\right\}$$

This can readily be changed into a surface integral. If $d\sigma$ denote an element of the surface of the vortex filament and cos α, cos β, cos γ, denote the direction cosines of the outward normal to this surface, we have

$$dx'dy'=d\sigma\cos\gamma,\ dx'dz'=d\sigma\cos\beta,$$
$$dy'dz'=d\sigma\cos\alpha;$$

therefore our integral becomes

$$\frac{1}{2\pi}\int\frac{1}{\sigma}\left[\xi'\cos\alpha+\eta'\cos\beta+\zeta'\cos\gamma\right]d\sigma$$

taken over the entire surface. Now from the equation

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0$$

we have also

$$\int\int\int\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)dx\,dy\,dz=0$$

by integration this becomes

$$\int\int \xi\, dy\, dz + \int\int \eta\, dx\, dz + \int\int \zeta\, dx\, dy = o$$

or as a surface integral,

$$\int (\xi \cos \alpha + \eta \cos \beta + \zeta \cos \gamma) d\sigma = o$$

consequently

$$\frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz} = o.$$

Substituting now the obtained values of U, V, W, in the equations $u = \frac{dW}{dy} - \frac{dV}{dz}$ &c... and we obtain for u, v, w, the following values:

$$u = -\frac{1}{2\pi}\int\int\int \frac{1}{r^{3}}, [\zeta'(y-y') - \eta'(x-x')] dx' dy' dz'$$

$$v = -\frac{1}{2\pi}\int\int\int \frac{1}{r^{3}}, [\xi'(z-z') - \zeta'(x-x')] dx' dy' dz'$$

$$w = -\frac{1}{2\pi}\int\int\int \frac{1}{r^{3}}, [\eta'(x-x') - \xi'(y-y')] dx' dy' dz'$$

or, as these may be expressed,

$$u = \frac{1}{2\pi}\int \left(\zeta \frac{d\frac{1}{r}}{dy} - \eta \frac{d\frac{1}{r}}{dz}\right) d\tau$$

$$v = \frac{1}{2\pi}\int \left(\xi \frac{d\frac{1}{r}}{dz} -- \zeta \frac{d\frac{1}{r}}{dx}\right) d\tau$$

$$w=\frac{1}{2\pi}\int\left(\eta\frac{d\frac{1}{r}}{dx}-\xi\frac{d\frac{1}{r}}{dy}\right)d\tau$$

Representing each of these differential expressions by u', v', w' respectively we see that u', v', w' are the increments of u, v, w, which correspond to the element $dx\ dy\ dz$ of the vortex filament. Writing for convenience of reference the equations

$$u'=\frac{d}{2\pi}\left(\zeta\frac{d\frac{1}{r}}{dy}-\eta\frac{d\frac{1}{r}}{dz}\right)$$

$$v'=\frac{d\tau}{2\pi}\left(\xi\frac{d\frac{1}{r}}{dz}-\zeta\frac{d\frac{1}{r}}{dx}\right)$$

$$w'=\frac{d\tau}{2\pi}\left(\eta\frac{d\frac{1}{r}}{dx}-\xi\frac{d\frac{1}{r}}{dy}\right)$$

we see that they give rise to the equation

$$\xi u'+\eta v'+\zeta w'=o.$$

This shows that, considering u', v', w' as the components of a certain new velocity, the direction of the resultant

$$\sqrt{u'^2+v'^2+w'^2}$$

of these components is at right angles angles to the direction of the axis of rotation of the element $d\tau$. Again, we have

$$u' \frac{dr}{dx} + v' \frac{dr}{dy} + w' \frac{dr}{dz} = o$$

and the direction of this resultant is also at right angles to the line r joining the element $d\tau$ to any other. Thus we see that each rotating element of the fluid mass implies in every other element a velocity whose direction is at right angles at the same time to the axis of rotation of the first and to the line joining the two elements—*i.e.*, at right angles to the plane containing the second element and the axis of rotation of the first. It is easily shown that

$$\sqrt{u'^2 + v'^2 + w'^2} = \frac{d\tau}{2\pi} \sqrt{\xi^2 + \eta^2 + \zeta^2} \, \frac{\sin \vartheta}{r^2}$$

when ϑ denotes the angle between r and the axis of rotation. From this equation we see that the magnitude of this induced velocity is directly proportional to the volume of the first element, its angular velocity and the sine of the angle between the line joining the two elements and the axis of rotation; and also inversely proportional to the square of the distance between the elements. Denote the angular velocities at the time $t=o$ by ξ_0, η_0, ζ_0 then the last equations of chapter I become

$$\xi=\xi_0\frac{dx}{da}+\eta_0\frac{dx}{db}+\zeta_0\frac{dx}{dc}$$

$$\eta=\xi_0\frac{dy}{da}+\eta_0\frac{dy}{db}+\zeta_0\frac{dy}{dc}$$

$$\zeta=\xi_0\frac{dz}{da}+\eta_0\frac{dz}{db}+\zeta_0\frac{dz}{dc}$$

a, b, c being the co-ordinates of an arbitrary particle we have for the co-ordinates of another situated indefinitely near this $a+da$, $b+db$, $c+dc$, and at the time t the co-ordinates will be x, y, z, $x+dx$, $y+dy$, $z+dz$; now suppose that at $t=o$ we have

$$\frac{da}{\xi_0}=\frac{db}{\eta_0}=\frac{dc}{\zeta^0}$$

that is da, db, cd proportional to the initial angular velocities—and suppose further that the direction of this indefinitely small line coincides with that of the axis of rotation. Call the common value of these ratios ε, ε being an indefinitely small quantity, independent of the time; then we have

$$da=\xi_0\varepsilon,\ db=\eta_0\varepsilon,\ dc=\zeta^{\bullet}\varepsilon,$$

Substituting these values in the above equations, and we have

$$da=\xi\varepsilon,\ dy=\eta\varepsilon,\ dz=\zeta z,$$

or

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta}$$

consequently the direction of the line joining the indefinitely near elements will at all times coincide with the direction of the axis of rotation. This combined with our definition of a vortex line shows us that every particle of fluid that lies on a vortex line at any instant will always remain there. If we call ω the resultant angular velocity we have

$$\omega=\sqrt{\xi^2+\eta^2+}\zeta=\overset{1}{\varepsilon}\sqrt{dx^2+dy^2+dz^2}$$

or the angular velocity so varies as to remain always proportional to the distance between the two particles. We have all along supposed the density of the fluid equal to unity. Remembering now our definition of a vortex filament, we see that any vortex filament must remain composed of the same fluid particles.

Calling now k the cross section of any filament and l an indefinitely small length of the filament, that is

$$l=\sqrt{dx^2+dy^2+dz^2}$$

we have kl=const.; but l is proportional to ω, therefore ωk=const. or the product of the angular velocity of an infinitely small portion of any filament into its cross section is a constant. Call now k_1, k_2 the cross sections of a filament at points whose angular velocities are given by ω_1, ω_2; and let $d\tau$ denote an element of the filament;

$$\int d\tau\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$$

$$=-\int d\sigma\left[\xi\cos(nx)+\eta\cos(ny)\right.$$

$$\left.+\zeta\cos(nz)\right]=-\int d\sigma\omega\cos(\omega n).$$

Now from the values of ξ, η, ζ, we see that the factor of $d\tau$ in the left hand integral is $=o$;

$$\therefore\int d\sigma\omega\cos(\omega n)=o$$

But at the ends of the portion of the vortex filament that we are considering, we have $\cos(\omega n) = \pm 1$, and for all other points $\cos(\omega n) = o$, and consequently our integral is equivalent to

$$\omega_1 k_1 - \omega_2 k_2 = o$$

or the product of the angular velocity and the cross section is a constant throughout the vortex filament. As each rotating element of the fluids implies rotation in every other element, we have that all the particles of fluid must be in motion, and consequently from the definition of vortex lines we see that these lines and consequently the vortex filaments cannot terminate within the fluid, but must either terminate in its surface or must return into themselves; the former of these cases is illustrated by the vortices formed in running water, and the latter by smoke rings.

If in the expression

$$\int\int\int d\tau \left\{ \frac{d(w\mathrm{V} - v\mathrm{W})}{dx} + \frac{d(u\mathrm{W} - w\mathrm{U}}{dy} + \frac{d(v\mathrm{U} - u\mathrm{V})}{dz} \right\}$$

we substitute for u, v and w their values in terms of the derivatives of U, V, W, we find that this is equal to zero, and from it we obtain the striking equation

$$\int d\tau\left\{u\left(\frac{dW}{dy}-\frac{dV}{dz}\right)+v\left(\frac{dU}{dz}-\frac{dW}{dx}\right)+w\left(\frac{dV}{dx}-\frac{dU}{dy}\right)\right\}$$

$$=\int d\tau\left\{U\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+V\left(\frac{du}{dz}-\frac{dw}{dx}\right)+W\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right\}$$

or

$$\int d\tau\,(u^2+v^2+w^2)=2\int\int\int d\tau\,[U\xi+V\eta+W\zeta]$$

But we may write for U, V, W the values

$$U=\frac{1}{2\pi}\int\frac{\xi' d\tau'}{r}$$

$$V=\frac{1}{2\pi}\int\frac{\eta' d\tau'}{r}$$

$$W=\frac{1}{2\pi}\int\frac{\zeta' d\tau'}{r}$$

$$\therefore\int d\tau\,(u^2+v^2+w^2)=\frac{1}{\pi}\int\frac{d\tau'}{r}\,[\xi'\xi'+\eta\eta'+\zeta\zeta']$$

But the expression for the energy of the fluid is

$$T=\tfrac{1}{2}\int d\tau(u^2+v^2+w^2)$$

and by virtue of the above we have

$$T=\frac{1}{2\pi}\int\int\frac{d\tau d\tau'}{r}[\xi\xi'+\eta\eta'+\zeta\zeta']$$

We can now take up the simplest case of vortex motion viz., that in which the motion is parallel to one plane. If we assume this plane as xy and further make the motion independent of z we have the angular velocities around x, and y, and the velocity in direction of z equal to zero, or

$$\xi=\eta=w=o$$

We have thus

$$u=\frac{dW}{dy},\ v=-\frac{dW}{dx},\ \zeta=\frac{dv}{dx}-\frac{du}{dy}$$

from which

$$-\zeta=\frac{d^2W}{dx^2}+\frac{d^2W}{dy^2}$$

which gives

$$W=-\frac{1}{\pi}\int\zeta\log\rho\, d\sigma$$

when $d\sigma$ is an element of the plane xy. Of course as ζ is independent of z we might have obtained this by integration from our general value of W before given. Here ρ represents the distance of the element $d\sigma$ in the plane xy from any other point in that plane. Each vortex filament implies in any other particle of the fluid a velocity whose components are,

$$-\frac{1}{\pi}\cdot\frac{\zeta d\sigma}{\rho}\cdot\frac{d\rho}{dy} \text{ and } \frac{1}{\pi}\cdot\frac{\zeta d\sigma}{\rho}\cdot\frac{d\rho}{dx}$$

and whose magnitude is

$$\frac{1}{\pi}\cdot\frac{\zeta d\sigma}{\rho}$$

The direction of this velocity is given by the cosines

$$-\frac{d\rho}{dy},\quad \frac{d\rho}{dx}$$

and of the line ρ by

$$\frac{d\rho}{dx},\quad \frac{d\rho}{dy}$$

or the direction of the velocity is at right angles to ρ. Assume two quantities $x_{\circ}$, $y_{\circ}$, which define by the equations

$$x_{\circ}\int\zeta d\sigma=\int x\zeta d\sigma,$$
$$y_{\circ}\int\zeta d\sigma=\int y\zeta d\sigma.$$

Now if we regard ζ as the density of a mass distributed over the element $d\sigma$ of the plane xy we see that x_0, y_0 will represent the co-ordinates of the center of gravity of this mass. Now l, the length of an indefinitely small portion of our vortex-filament cannot alter, consequently by virtue of the equation kl=constant, k or $d\sigma$ cannot vary with the time and by virtue of $k\omega$=constant, ω or ζ cannot vary with respect to the time; consequently, we have, by differentiating with reference to t

$$\frac{dx_0}{dt}\int \zeta d\sigma = \int \frac{dx}{dt} \zeta d\sigma$$

$$\frac{dy_0}{dt}\int \zeta d\sigma = \int \frac{dy}{dt} \zeta d\sigma$$

Substituting

$$\frac{dx}{dt} = u = -\frac{1}{\pi}\int \frac{\zeta' d\sigma'}{\rho}, \frac{y-y'}{\rho'}$$

$$\frac{dy}{dt} = v = \frac{1}{\pi}\int \frac{\zeta' d\sigma'}{\rho}, \frac{x-x'}{\rho}$$

we have

$$\frac{dx_0}{dt}\int \zeta d\sigma = -\frac{1}{\pi}\int\int \rho\rho' d\sigma d\sigma' \frac{y-y'}{\rho^2}$$

$$\frac{dy^0}{dt}\int \zeta d\sigma = \frac{1}{\pi}\int\int \rho\rho' d\sigma d\sigma' \frac{x-x'}{\rho^2}$$

The double integrals are$=o$ therefore

$$\frac{dx_0}{dt}=o \quad \frac{dy_0}{dt}=o$$

or the center of gravity of the filament does not change with the time. In the case of only one vortex filament let us write

$$\int\zeta d\sigma=m$$

for particles as at finite distance from the filament we have

$$u=\frac{dW}{dy} \quad v=-\frac{dW}{dx}, \quad W=-\frac{1}{\pi}m\log\rho$$

for particles indefinitely near the filament we see that W, u, v, are infinite and depend upon the cross section of the filament and the angular velocity ρ. We also know that at the center of gravity of the filament u and $v=o$. Each particle of fluid that is at a finite distance from the filament we see has a uniform velocity of $\frac{m}{\pi\rho}$ and moves in a circle whose center is the center of gravity of the vortex filament. Suppose we now assume a number of filaments whose cross section is indefinitely small. Write in general

$$\int \zeta_i \, d\sigma_i = m_i$$

and let x_i y_i denote the co-ordinates of the centers of gravity of the filaments at the time t and ρ_i their distances from the point (xy). Then for all points at finite distances from the filaments we have as before

$$u = \frac{dW}{dy}, \; v = -\frac{dW}{dx}, \; W = -\frac{1}{\pi}\Sigma m_i \log \rho_i$$

Now each filament inducing a certain amount of motion in every other particle of the fluid induces motion in the centers of gravity of every other filament, therefore the filaments change their places in the fluid. But here, as before, that portion of u, v which each vortex filament gives to its own center of gravity is $=o$. Suppose that the point from which the ρ_s are measured is one of the centers of gravity, for example x y. This will materially simplify the investigation by confining us exclusively to the influence of the system of vortices upon its different members and as this point x, y, is an arbitrary point no generality is lost. We have thus

$$u_1 = \frac{dW_1}{dy_1} \quad v_1 = -\frac{dW_1}{dx_1}$$

$$W_1 - = \frac{1}{\pi}$$

$$(m_2 \log \rho_{12} + m_3 \log \rho_{13} + \ldots m_i \log \rho_{1i})$$

or briefly

$$W_1 = -\frac{1}{\pi} \sum_{i=2}^{i} m_i \log \rho_{1i}.$$

We can now assume a function Q such that

$$Q = -\frac{1}{\pi} \Sigma m_i m_j \log \rho_{ij}$$

and then we have

$$m_1 \frac{dx_1}{dt} = \frac{dQ}{dy_1}, \; m_2 \frac{dx_2}{dt} = \frac{dQ}{dy_2}, \ldots\ldots$$

$$m_1 \frac{dy_1}{dt} = -\frac{dQ}{dx_1}, \; m_2 \frac{dy_2}{dt} = -\frac{dQ}{dy_2}, \ldots\ldots$$

A complete system of integrals cannot be in general obtained, but by observing one peculiarity of Q we can obtain two integrals—

$$\rho_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

If we increase x_i, x_j, or y_i, y_j, by the same quantity, ρ_{ij} will be unaltered; this gives us then

$$\Sigma \frac{dQ}{dx_i}=o, \ \Sigma \frac{dQ}{dy_i}=o$$

or

$$\Sigma m_i \frac{dy_i}{dt}=o, \ \Sigma m_i \frac{dx_i}{dt}=o$$

from which

$$\Sigma m_i x_i = \text{const.}$$

$$\Sigma m_i y_i = \text{const.}$$

or the center of gravity of the system of vortex filaments is unaltered. Again since

$$m_i \left(\frac{dx_1}{dt} dy_i - \frac{dy_i}{dt} dx_i \right)=o$$

we have

$$dQ=o, \ \therefore \ Q=\text{const.}$$

the equation of the lines of flow of the fluid. Introduce now polar co-ordinates,

$$x_1=\rho_1 \cos \vartheta_1 \quad x_2=\rho_2 \cos \vartheta_2 + \ldots .$$
$$y_1=\rho_1 \sin \vartheta_1 \quad y_2=\rho_2 \sin \vartheta_2 + \ldots .$$

we have by these substitutions

$$m_1\rho_1 \frac{d\rho_1}{dt} = \frac{dQ}{d\vartheta_1}, \ m_2\rho_2 \frac{d\rho_2}{dt} = \frac{dQ}{d\vartheta_2} \ldots \ldots$$

$$m_1\rho_1 \frac{d\vartheta_1}{dt} = -\frac{dQ}{d\rho_1}, \ m_2\rho_2 \frac{d\vartheta_2}{dt} = -\frac{dQ}{d\rho_2} \ldots$$

If now we increase all the ϑ's by the same quantity, Q will evidently remain unaltered and we have the equation,

$$\Sigma \frac{dQ}{d\vartheta_i} = o$$

The first row now gives by addition

$$\Sigma m_i \frac{d\rho_i}{dt} = o$$

or $$\Sigma m_i \rho_i{}^2 = \text{const.}$$

Now let us suppose d to remain unchanged but ρ to become $n\rho$, then log ρ becomes log ρ + log n, and log ρ_{ij} becomes log ρ_{ij} 1 + log n, and in consequence Q will be increased by

$$-\frac{1}{\pi}[m_1 m_2 \log n + m_1 m_3 \log n + m_2 m_3 \log n + \ldots . m_j n_j \log n +]$$

or

$$-\frac{1}{\pi} \log n \, \Sigma \, m_i \, m_j$$

and consequently we have

$$\Sigma \frac{d\,Q}{d \log \rho_i} = -\frac{1}{\pi} \Sigma \, m_i \, m_j$$

or

$$\Sigma \rho_i \frac{dQ}{d\rho_i} = -\frac{1}{\pi} \Sigma \, m_i \, m_j$$

But we have also

$$\frac{dQ}{d\rho_i} = - m_i \, \rho_i \, \frac{d\vartheta_i}{dt}$$

Substituting this value gives

$$\Sigma \, m_i \, \rho_i^2 d\vartheta_i = \frac{dt}{\pi} \, \Sigma m_i \, m_j$$

Assume now the case of only these vortex filaments existing in the fluid.

The equations $\Sigma m_i x_i =$ const.; and $\Sigma m_i y_i =$ const. become

$$m_1\rho_1\cos\vartheta_1 + m_2\rho_2\cos\vartheta_2 + m_3\rho_3\cos\vartheta_3 = C_1$$
$$m_1\rho_1\sin\vartheta_1 + m_2\rho_2\sin\vartheta_2 + m_3\rho_3\sin\vartheta_3 = C_2$$

Multiply the first equation by cos ϑ, the second by sin ϑ and add,

$$m_1\rho_1 + m_2\rho_2\cos(\vartheta_2 - \vartheta_1) + m_3\rho_3\cos(\vartheta_3 -{}_1\vartheta)$$
$$= C_1\cos\vartheta_1 + C_2\sin\vartheta_1$$

Again multiply the first by sin ϑ, and the second by cos ϑ, and add

$$m_2\rho_2\sin(\vartheta_2 - \vartheta_1) + m_3\rho_3\sin(\vartheta_3 - \vartheta_1)$$
$$= C_2\cos\rho_2 - C_1\sin\vartheta_1$$

Again,

$$Q = -\frac{1}{\pi}[m_1 m_2 \log\rho_{12} + m_1 m_0 \log\rho_{13}$$
$$+ m_2 m_0 \log\rho_{23} + \ldots\ldots] = \text{const.}$$

and

$$m_1\rho_1^2 + m_2\rho_2^2 + m_3\rho_3^2 = \text{const.}$$

Through these four equations we may express any four of the quantities ρ_1, ρ_2, ρ_3, $\vartheta_2 - \vartheta_1$, $\vartheta_3 - \vartheta_1$ in terms of the fifth; for example ρ_1; then the equations

$$m_1 \rho_1 d\vartheta_1 = -\frac{dQ}{d\rho_1} dt$$

and

$$\sum_1^3 {}_i m_i \rho_i^2 d\vartheta_i = \frac{dt}{\pi} \sum_1^3 m_i m_j$$

will enable us to express ϑ_1 and t as functions of ρ_1 and afford a complete solution of the problem.

Assume now only two vortices, and take the origin of co-ordinates at their common center of gravity. This point does not move, and we have

$$\frac{d\vartheta_1}{dt} = \frac{d\vartheta_2}{dt}$$

Q in this case becomes

$$-\frac{1}{\pi} m_1 m_2 \log (\rho_1 + \rho_2) = \text{const.}$$

and also

$$\Sigma m_i \rho_i^2 = m_i \rho_i^2 + m_i \rho_i^2 = \text{const.}$$

from these two equations we obtain

$$\rho_1 = \text{const.}, \quad \rho_2 = \text{const.}$$

Again the equation

$$\Sigma m_i\rho_2^2 d\vartheta_1 = \frac{dt}{\pi}\,\Sigma m_i m_j$$

becomes

$$m_1\rho_1^2 d\vartheta_1 + m_2\rho_2^2 d\vartheta_2 = \frac{dt}{\pi}\,.\,m_1 m_2$$

giving

$$\frac{d\vartheta_1}{dt} = \frac{d\vartheta_2}{dt} = \frac{1}{\pi}\cdot\frac{m_1 m_2}{m_1\rho_1^2 + m_2\rho_2^2}.$$

If the direction of rotation of both vortex filaments is the same $m_1 m_2$ [which depend on ζ_1 and ζ_2] have the same sign. But suppose $m_1 = -m_2$, then

$$\frac{d\vartheta_1}{dt} = \frac{1}{\pi}\,\frac{m_1}{\rho_2^2 - \rho_1^2}$$

But we have now for the center of gravity

$$x = \frac{m_1x_1 - m_1x_2}{m_1 - m_1} = \infty$$

$$y = \frac{m_1y_1 - m_1y_2}{m_1 - m_1} = \infty$$

or the center of gravity of the two filaments lies at infinity. Their velocities = their angular velocity x by the distance from the center of gravity differ from each other by an infinitely small quantity and can be expressed by

$$\frac{\rho_1+\rho_2}{2}\frac{d\theta_1}{dt}$$

but by our preceding equation giving the value of $\frac{d\theta_1}{dt}$ this is

$$=\frac{1}{2\pi}\frac{m_1}{\rho_2-\rho_1}$$

the direction of the motion is, of course, perpendicular to the line giving the centers of gravity of the two filaments. The particles of fluid lying between the filaments move forwards in the same direction as do the filaments, the one-half way between them moving four times as fast. Let us suppose that the vortex filaments at the beginning of the motion lie on the axis of x at equal distances from the origin, then the particle above referred to will lie at the origin. Also write $\frac{\rho_2-\rho_1}{z}=a$ the absolute distance of each filament from the origin. We have then for the co-ordinates of the filaments at the time t (a, y') and, by virtue of what has been said the co-ordinates of the particle originally at the origin will be

$(0,4y')$. The equations of the lines joining these two points are

$$\begin{vmatrix} x, & y\,, & 1 \\ o, & 4y', & 1 \\ \pm a, & y', & 1 \end{vmatrix} = o$$

The intersections of these lines with $y=o$ are given by

$$x=\pm\frac{4a}{3} \text{ or } x=\pm\tfrac{4}{3}(\rho_2-\rho_1)$$

that is, the lines joining the particle half way between the filaments with the centers of gravity of the same pass through fixed points on the line joining the original positions of the centers of gravity of the filaments. The points lie outside of the original positions of these centers of gravity and at an absolute distance from them$=\tfrac{1}{3}(\rho_2-\rho_1)=\tfrac{2}{3}, \frac{\rho_3-\rho_1}{z}$.

The particles of fluid that lie in the plane bisects at right angles the line joining the two vortex filaments will remain in this plane. If this plane be considered as a fixed boundary, we have by considering one of the filaments the case of a filament moving parallel to a fixed plane which limits the extent of the fluid.

Let us now assume that the vortex filaments are so arranged as to form the continuous surface of an elliptic cylinder of finite cross section, and further assume ζ as constant for every point of this cross section. As the same particles of fluid constantly remain in any vortex filament, the bounding ellipse of the cross section of the fluid will always be composed of the same fluid particles. The equation of this line will be a function of x, y and t, and may be written for the moment as

$$f(x,Y,t)=o.$$

Then by virtue of the above we have generally

$$\frac{df}{dt}+\frac{df}{dx}\frac{dx}{dt}+\frac{df}{dy}\frac{dy}{dt}=o$$

or

$$\frac{df}{dt}+u\frac{df}{dx}+v\frac{df}{dy}=o$$

But,

$$u=\frac{dW}{dy},\ v=-\frac{dW}{dx}.$$

Therefore this equation becomes

$$\frac{df}{dt}+\frac{dW}{dy}\frac{df}{dx}-\frac{dW}{dx}\frac{df}{dy}=o.$$

Now the general equation of our ellipse is

$$f=ax^2+2\beta xy+\gamma y^2-1$$

when a, β, γ are functions of t. Assume another system of co-ordinates coinciding with the axes of the ellipse and also passing through its center; then

$$x'=x\cos\theta+y\sin\theta$$
$$y'=-x\sin\theta+y\cos\theta.$$

Call a and b the semi-axes of the ellipse then we have for its equation,

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1.$$

Substituting for x', y', their values as given in terms of x and y, and this becomes,

$$x^2\frac{b^2\cos^2\vartheta+a^2\sin^2\vartheta}{a^2b^2}+2xy\frac{[(b^2-a^2)\cos\vartheta\sin\vartheta]}{a^2b^2}$$
$$+y^2\frac{b^2\sin^2\vartheta+a^2\cos^2\vartheta}{a^2b^2}=1$$

Comparing this with

$$ax^2+2\beta xy+\gamma y^2=1$$

and we obtain,

$$a^2b^2\alpha = b^2\cos^2\vartheta + a^2\sin^2\vartheta$$
$$a^2b^2\beta = (b^2-a^2)\cos\vartheta\sin\vartheta$$
$$a^2b^2\gamma = b^2\sin^2\vartheta + a^2\cos^2\vartheta$$

In these a and b are constant, but α, β, γ, ϑ are functions of t. Now W satisfies the equation

$$\frac{d^2W}{dx^2} + \frac{d^2W}{dy^2} = -\zeta$$

for all points in the interior of the ellipse, and its value is obtained by integration of the equation

$$\frac{\zeta}{2}ab\int_0^\infty \frac{1-\frac{x^2}{a^2+\lambda}-\frac{y^2}{b^2+\lambda}}{\sqrt{(a^2+\lambda)\,(b^2+\lambda)}}\,.\,d\lambda$$

The integral of this is

$$\frac{\zeta}{2}ab\left(2\log\frac{1}{a+b}+1\right)-\frac{\zeta}{a+b}\,(bx'^2+ay'^2)$$

or, since we only use the derivatives of W, we may write it,

$$W = C - \frac{\zeta}{a+b}\,(bx'^2+ay'^2)$$

for all interior points, and for points at the boundary. If now we write for brevity,

$$\mathrm{A}=b\cos^2\vartheta+a\sin^2\vartheta$$
$$\mathrm{B}=(b-a)\cos\vartheta\sin\vartheta$$
$$\Gamma=b\sin^2\vartheta+a\cos^2\vartheta$$

we have

$$\mathrm{W}=\mathrm{C}-\frac{\delta}{a+b}(\mathrm{A}x^2+2\mathrm{B}xy+\Gamma y^2).$$

Let us examine again the condition that we obtained for the bounding line of the cross section of the cylinder, viz:

$$\frac{df}{dt}+\frac{df}{dx}\frac{d\mathrm{W}}{dy}-\frac{df}{dy}\frac{d\mathrm{W}}{dx}=o$$

we have

$$\frac{df}{dt}=\frac{d\alpha}{dt}x^2+2\frac{d\beta}{dt}xy+\frac{d\vartheta}{dt}y^2$$

$$\frac{df}{dx}\frac{d\mathrm{W}}{dy}=-(\alpha x+\beta)(\vartheta\mathrm{B}x+\Gamma y)\frac{4\delta}{a+b}$$

$$-\frac{df}{dy},\frac{d\mathrm{W}}{dx}=(\beta x+\vartheta y)(\mathrm{A}x+\mathrm{B}y)\frac{4\delta}{a+b}.$$

Equating to zero the co-efficients of x^2 xy, and y^2 separately we have

$$(a+b)\frac{d\alpha}{dt}=4\delta(\alpha\mathrm{B}-\beta\mathrm{A}),$$

$$(a+b)\frac{d\beta}{dt}=2\delta(\alpha\Gamma-\gamma\mathrm{A}),$$

$$(a+b)\frac{d\gamma}{dt}=4\delta(\beta\Gamma-\gamma\mathrm{B}).$$

If ϑ cannot be determined as such a function of t as to satisfy these equations then will our equation of condition

$$\frac{df}{dt}+\frac{df}{dx}\frac{dW}{dy}-\frac{df}{dy}\frac{dW}{dx}=o$$

hold for all points in the cross section. Forming the derivatives of α, β, γ with respect to t and transforming them by reference to the values of A, B, Γ we find that the function of t sought is given by the equation

$$\frac{d\vartheta}{dt}=2\,\zeta\frac{ab}{(a+b)^2}$$

We have thus the value of the angular velocity with which the cylinder rotates around its axis. The rotation of the cylinder also induces relative motions among the component vortex filaments. These are obtained by regarding x' and y' as functions of t. We have by differentiation

$$\frac{dx'}{dt}=y'\,\frac{d\vartheta}{dt}\quad\frac{dy'}{dt}=-x'\,\frac{d\vartheta'}{dt}$$

and the other components of the velocities in the directions of x' and y' are

$$\frac{dW}{dy'}, -\frac{dW}{dx'}$$

Therefore we have by combining these

$$\frac{dx'}{dt}=\frac{dW}{dy'}+y'\frac{d\vartheta}{dt}$$

$$\frac{dy'}{dt}=-\frac{dW}{dx'}-x'\frac{d\vartheta}{dt}$$

Now,

$$\frac{dW}{dy'}=-\frac{2a\zeta}{a+b}y', \text{and}, \frac{dW}{dx'}=-\frac{2\zeta bx'}{a+b}$$

and

$$\frac{d\vartheta}{dt}\zeta=2\frac{ab}{(a+b)^2}.$$

Therefore

$$\frac{dx'}{dt}=-\frac{2\zeta a^2}{(a+b)^2}y', \frac{dy'}{dt}=\frac{2\zeta b^2}{(a+b)^2}x'.$$

Differentiate each of these for t and with

$$\theta=2\zeta\frac{ab}{(a+b)^2}$$

and we have after integration of the resulting well-known powers

$$x'=a\,l\cos(\theta t+i)$$

$$y'=b\,l\cos(\theta t+i)$$

when l and i are the constants of integration and determine the particle of fluid to which x' y' have reference. $\rho >$ 1 because then for the cases when $\cos(\theta t+i)=1$ we should have $x' > a$ which cannot be, $\therefore$ l is a proper fraction. θ of course denotes the angular velocity of the cylinder, or

$$\frac{d\vartheta}{dt}=\theta$$

from which $d=\theta t$.

Solving the equations

$$x'=x\cos\vartheta+y\sin\vartheta$$
$$y'=-x\sin\vartheta+y\cos\vartheta$$

for x and y gives

$$x=x'\cos\vartheta-y'\sin\vartheta$$
$$y=x'\sin\vartheta+y'\cos\vartheta$$

Substituting for x', y' and ϑ their values these become

$$x=al\cos(\theta t+i)\cos\theta t$$
$$-bl\sin(\theta t+i)\sin\theta t$$

$$y=al\cos(\theta t+i)\sin\theta t$$
$$+bl\sin(\theta t+i)\cos\theta t$$

by expanding the quantities cos sin $(\theta t+i)$ and collecting the terms these equations may be written.

$$x = \frac{a+b}{2} l \cos (2\theta t + i) + \frac{a-b}{2} l \cos i$$

$$y = \frac{a+b}{2} l \sin (2\theta t + i) - \frac{a-b}{2} l \sin i$$

by differentiation with respect to t we obtain

$$\frac{dx}{dt} = -(a+b)\theta l \sin (2\theta t + i)$$

$$\frac{dy}{dt} = (a+b)\theta l \cos (2\theta t + i)$$

from which

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = (a+b)\theta l = 2\zeta l. \frac{ab}{a+b}$$

and also,

$$\left(x - \frac{a-b}{2} l \cos i\right)^2 + \left(y + \frac{a-b}{2} l \sin i\right)^2 = \left(\frac{a+b}{2}\right)^2 l^2$$

From these equations we see that each vortex filament moves in a circle with uniform angular velocity, the time of rotation being evidently $\frac{\pi}{\theta}$; and that the position of the center and the radius of the circle varies for different filaments.

Suppose one of the semi-axes a or b to be infinitely greater than the other; this gives $\theta=o$, and consequently $\vartheta=o$, or the straight line which has become the limiting case of the ellipse does not rotate. If $a=b$ our ellipse becomes a circle, and we have $\theta=\frac{\zeta}{2}$; in this case there is no change of the relative positions of the vortex filaments, but they all rotate around the common central axis with angular velocity ζ.

The next case of vortex motion that we shall consider is that in which the vortex lines are circles having their centers in the axis of z. The direction of the axis of rotation of each fluid particle will lie in a plane at right angles to z and be parallel to the tangent to the vortex line at that point. The reader will do well to observe the motion of a ring of tobacco smoke; he will see that the ring seems to be turning inside out, each particle moving in a plane passing through the axis of the ring and revolving in a circle whose axis is in the direction of the tangent to the ring. He will also observe that there is no motion around the axis of the ring. Now if we introduce polar co-ordinates ρ and ϑ we have, evidently,

$$x=\rho\cos\vartheta \quad y=\rho\sin\vartheta$$

and for the rotations we may obviously write

$$\xi=-\lambda\sin\vartheta,\ \eta=\lambda\cos\vartheta,\ \zeta=o,$$

when λ is not a function of ϑ. The equation of the path of the particle is evidently one between x, y, z where x and y are connected by the relation

$$\rho=\sqrt{x^2+y^2}$$

so that the equation of the path can be made to depend only upon ρ and z. Resuming now our equations

$$u=\frac{dW}{dy}-\frac{dV}{dz},\ v=\frac{dU}{dz}-\frac{dW}{dx},$$
$$w=\frac{dV}{dx}-\frac{dU}{dy}$$

and observing that

$$W=\frac{1}{2\pi}\int\frac{\zeta d\tau}{r}$$

these give

$$W=o,\ u=-\frac{dV}{dz},\ v=\frac{dU}{dz}\quad w=\frac{dV}{dx}-\frac{dU}{dy}$$

If now we assume an element $d\tau'$ given by the co-ordinates ϑ', ρ', z' and for which λ becomes λ'; also denote by r its distance from the point (ρ, ϑ, z) or $(x\, y\, z)$. Now our equations for U and V are

$$U=\frac{1}{2u}\int\frac{\xi' d\tau'}{r}=-\frac{1}{2\pi}\int\frac{\lambda' \sin\ \vartheta' d\tau'}{r}$$

$$V=\frac{1}{2\pi}\int\frac{\eta' d\tau'}{r}=\quad\frac{1}{2\pi}\int\frac{\lambda' \cos\ \vartheta' d\tau'}{r}$$

When

$$d\tau'=dx'\ dy'\ dz'=\rho' d\rho'\ dz'\ d\vartheta'$$

and

$$r=\sqrt{(z'-z)^2+\ \rho^2+\rho'^2-2\rho\rho'\ \cos\ (\vartheta'-\vartheta)}$$

Now make for convenience

$$\vartheta'-\vartheta=\varphi$$

this gives $\vartheta'=\varphi+\vartheta$ and $d\vartheta'=d\varphi$ therefore we have again for U and V,

$$U=-\frac{1}{2\pi}\int\rho' d\rho'\int dz'\int$$

$$\frac{\lambda'[\sin\ \varphi\ \cos\ \vartheta+\cos\varphi\ \sin\ \vartheta]d\varphi}{\sqrt{(z'-z)^2+\rho^2+\rho'^2-2\rho\rho'\ \cos\ \varphi}}$$

$$V=\frac{1}{2\pi}\int\rho' d\rho'\int dz'\int\frac{\lambda'\,[\sin\varphi\sin\vartheta-\cos\varphi\cos\vartheta]\,d\varphi}{\sqrt{(z'-z)^2+\rho^2+\rho'^2-2\rho\rho'\cos\varphi}}$$

In both cases we have the integral

$$\int\frac{\sin\varphi\,d\varphi}{\sqrt{(z'-z)^2+\rho^2+\rho'^2-2\rho\rho'\cos\varphi}}$$

this is to be taken between o and 2π, these being the limits of $\vartheta'-\vartheta$,

$$\int_o^{2\pi}\frac{\sin\varphi\,d\varphi}{\sqrt{(z'-z)^2+\rho^2+\rho'^2-2\rho\rho'\cos\varphi}}$$

$$=\log\sqrt{(z'-z)^2+\rho^2+\rho'^2-2\rho\rho'\cos\varphi}\Big]_o^{2\pi}$$

$$=\frac{1}{2}\log\frac{(z'-z)^2+\rho^2+\rho'^2-2\rho\rho'\cos\varphi}{(z'-z)^2+\rho^2+\rho'^2-2\rho\rho'\cos\varphi}=o$$

and our expressions are thus reduced to

$$U=-\frac{1}{2\pi}\int\rho' d\rho'\int dz'\int\frac{\lambda'\cos\varphi\sin\vartheta\,d\varphi}{\sqrt{(z'-z)^2+\rho^2+\rho'^2-2\rho\rho'\cos\varphi}}$$

$$V=\frac{1}{2\pi}\int\rho' d\rho'\int dz'\int\frac{\lambda'\cos\varphi\cos\vartheta\,d\varphi}{\sqrt{(z'-z)^2+\rho^2+\rho'^2-2\rho\rho'\cos\varphi}}$$

Denoting the common integral by Φ we have

$$U = -\frac{1}{2\pi}\int \rho' d\rho' \int dz' \, \lambda' \, \Phi \sin \vartheta$$

$$V = \frac{1}{2\pi}\int \rho' d\rho' \int dz' \, \lambda' \, \Phi \cos \vartheta$$

Φ will clearly be of the form

$$\Phi(z'-z, \rho, \rho').$$

U and V now differ only by a factor, in fact we have

$$V = -U \tan \vartheta.$$

So if we write

$$P = \frac{1}{2\pi}\int\int \rho' d\rho' dz' \lambda' \Phi$$

we will have briefly

$$U = -P \sin \vartheta,$$
$$V = P \cos \vartheta.$$

The value of the function Φ is not difficult to obtain; we have

$$\Phi = \int_0^{2\pi} \frac{\cos \varphi \, d\varphi}{\sqrt{(z'-z)^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos\varphi}}$$

We will find the integration much simplified by the introduction of a new variable Ψ defined by the equation

$$\Psi=\frac{\pi-\varphi}{2}$$

Then we have

$$\cos\varphi=-[1-2\sin^2\psi]$$
$$\cos\varphi d\varphi=2[1-2\sin^2\psi]d\psi$$

and for the limits we have when $\varphi=(o,2\pi),\psi=\left(\frac{\pi}{2},-\frac{\pi}{2}\right)$. Making this transformation we have

$$\Phi=2\int_{+\frac{\pi}{2}}^{-\frac{\pi}{2}}\frac{[1-2\sin^2\psi]d\psi}{\sqrt{(z'-z)^2+\rho^2+\rho'^2+2\rho\rho'-4\rho\rho\sin^2\psi}}$$

or,

$$\Phi=-4\int_o^{\frac{\pi}{2}}\frac{[1-2\sin^2\Psi]d\Psi}{\sqrt{(z'-z)^2+(\rho+\rho')^2-4\rho\rho'\cos\varphi}}$$

$$=-4\int_o^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{(z'-z)^2+(\rho'+\rho)^2}\sqrt{1-\frac{4\rho\rho'\sin\Psi}{(z'-z)^2+(\rho'+\rho)^2}}}$$

$$+4\int_{0}^{\frac{\pi}{2}}\frac{2\sin^2\psi\,d\psi}{\sqrt{(z'-z)^2+(\rho'+\rho)^2}\sqrt{1-\frac{4\rho\rho'\sin^2\Psi}{(z'-z)^2+(\rho'+\rho)^2}}}$$

Make,

$$\frac{4\rho\rho'}{(z'-z)^2+(\rho'+\rho)^2}=k^2$$

then

$$\frac{1}{\sqrt{z'-z)^2+(\rho'+\rho^2)}}=\frac{k}{2\sqrt{\rho\rho'}}$$

then

$$\Phi=-\frac{2k}{\sqrt{\rho\rho'}}\int_{0}^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

$$+\frac{2k}{\sqrt{\rho\rho'}}\int_{0}^{\frac{\pi}{2}}\frac{2\sin^2\psi\,d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

The first of these is the complete elliptic integral of the first kind; we will, as usual, denote it by K. Examine now the second integral; we have on multiplying it numerator and denominator by k

$$\frac{4}{k\sqrt{\rho\rho'}}\int_0^{\frac{\pi}{2}}\frac{k^2\ \sin^2\psi d\phi}{\sqrt{1-k^2\sin^2\Psi}}$$

$$=\frac{4}{k\sqrt{\rho\rho'}}\int_0^{\frac{\pi}{2}}\frac{1-(1-k^2\sin^2\psi)}{\sqrt{1-k^2\sin^2\psi}}d\psi$$

$$=\frac{4}{k\sqrt{\rho\rho'}}\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-k^2\ \sin^2\ \psi}}$$

$$-\frac{4}{k\sqrt{\rho\rho'}}\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\ \sin^2\ \psi}d\psi$$

The second of these is the elliptic integral of the second kind and denoted by E; we have then finally for Φ,

$$\Phi=-\ \frac{2k}{\sqrt{\rho\rho'}}\ \mathrm{K}+\frac{4}{k\sqrt{\rho\rho'}}\mathrm{K}-\frac{4}{k\sqrt{\rho\rho'}}\ \mathrm{E}$$

or

$$\Phi=\frac{2}{\sqrt{\rho\rho'}}\left\{\left(\frac{2}{k}-k\right)\mathrm{K}-\frac{2}{k}\,\mathrm{E}\right\}$$

and consequently

$$P=\frac{1}{\pi}\int\int\lambda'\sqrt{\frac{\rho'}{\rho}}\left\{\left(\frac{2}{k}-k\right)K-\frac{2}{k}E\right\}d\rho' dz',$$

$$U=-\frac{1}{\pi}\int\int\lambda'\sqrt{\frac{\rho'}{\rho}}\left\{\left(\frac{2}{k}-k\right)K-\frac{2}{k}E\right\}\sin\vartheta\, d\rho' dz',$$

$$V=\frac{1}{\pi}\int\int\lambda'\sqrt{\frac{\rho'}{\rho}}\left\{\left(\frac{2}{k}-k\right)K-\frac{2}{k}E\right\}\cos\vartheta\, d\rho' dz'.$$

In the function that we have denoted by Φ we see that the derivatives taken for z and z' have the same absolute values but opposite signs; consequently

$$\int\int\lambda\lambda'\frac{d\Phi}{dz}\rho\rho' d\sigma\, d\sigma'=o$$

when $d\sigma=d\rho dz$. But we have also

$$\frac{dP}{dz}=\frac{1}{2\pi}\int\lambda'\rho'\frac{d\Phi}{dz}d\sigma'$$

therefore

$$\int\rho\frac{dP}{dz}\lambda d\sigma=o.$$

But we have for the energy T the equation

$$T=\int d\tau(U\xi+V\eta)$$

Substituting for U,V,ξ and η their values this becomes,

$$T = \int P\lambda\, d\tau = \iiint P\lambda\,\rho\, d\rho\, dz\, d\vartheta$$

integrating with respect to ϑ from o to 2π, and writing again $d\rho dz = d\sigma$,

$$T = 2\pi \int P\rho\lambda d\sigma.$$

Substituting for P its value we have

$$T = \int \Phi\rho\rho'\lambda\lambda'\, d\sigma\, d\sigma'$$

when of course $d\sigma$ and $d\sigma'$ denote the cross sections of the vortex filaments under consideration. Let S denote the component of velocity in which ρ increases:

$$S = \sqrt{u^2 + v^2}$$

which is the same as

$$u = S \cos \vartheta \qquad v = S \sin \vartheta.$$

But

$$u = -\frac{dV}{dz} = -\frac{dP}{dz} \cos \vartheta$$

$$v = \frac{dU}{dz} = -\frac{dP}{dz} \sin \vartheta$$

therefore

$$S = -\frac{dP}{dz}.$$

Now for w we have

$$w=\frac{dV}{dx}-\frac{dU}{dy}$$

and substituting for U and V their values.

This gives

$$w=\frac{dP}{d\rho}+\frac{P}{\rho}$$

or

$$w\rho=\frac{d(P\rho)}{d\rho}$$

and we may also write

$$s\rho=-\frac{d(P\rho)}{dz} \text{ since } \frac{d\rho}{dz}=o.$$

From these we have the equation $\int\rho\frac{dP}{dz}\lambda\,d\sigma=o$ in the form

$$\int\rho s\lambda d\sigma=o$$

and also

$$s=\frac{d\rho}{dt} \text{ and } w=\frac{dz}{dt}$$

therefore

$$\int\rho\frac{d\rho}{dt}\lambda d\sigma=o$$

and since for each vortex filament $\lambda d\sigma$ is constant this gives

$$\int\rho^2\lambda d\sigma=\text{const.}$$

Some other interesting forms may be given before we proceed to the examination of a special case. We had

$$k^2 = \frac{4\rho'\rho}{(z'-z)^2+(\rho'+\rho)^2}$$

Taking logarithms this becomes

$$2 \log k = \log \rho\rho' - \log \left[(z'-z)^2+(\rho'+\rho)^2\right]$$

Differentiating with respect to ρ and z we obtain

$$\frac{2}{k}\rho\frac{dk}{d\rho} = \frac{(z'-z)^2+\rho'^2-\rho^2}{(z'-z)^2+(\rho'+\rho)^2}$$

$$\frac{2}{k}z\frac{dk}{dz} = \frac{zz(z'-z)}{(z'-z)^2+(\rho'+\rho)^2}$$

consequently,

$$\frac{2}{k}\left\{\rho\frac{dk}{d\rho}+z\frac{dk}{dz}\right\} = \frac{z'^2-z^2+\rho'^2-\rho^2}{(z'-z)^2+(\rho'+\rho)^2}$$

The denominator of this second member does not change by the interchange of accented and unaccented letters, but the numerator does change its sign, also k does not change by making the same transfer, therefore

$$\rho\frac{dk}{d\rho}+z\frac{dk}{dz}$$

assumes the opposite value by writing the accented letters in the place of the unaccented and vice versa. If from the value of Φ in terms of K and E we obtain $\frac{d\Phi}{dk}$ we will see that this quantity does not alter by the interchange of accented and unaccented letters, consequently the quantity

$$\frac{d\Phi}{dk}\left(\rho\frac{dk}{d\rho}+z\frac{dk}{dz}\right)$$

assumes the opposite value after the interchange. We have by partial differentiation of Φ

$$\frac{d\Phi}{d\rho}=\frac{d\Phi}{d\rho}+\frac{d\Phi}{dk}\frac{dk}{d\rho}$$

$$=\frac{d\Phi}{dk}\cdot\frac{dk}{d\rho}-\frac{1}{z}\frac{\Phi}{\rho}$$

$$\frac{d\Phi}{dz}=\frac{d\Phi}{dk}\frac{dk}{dz}$$

Consequently the above quantity assumes the form

$$\rho\frac{d\Phi}{d\rho}+z\frac{d\Phi}{dz}+\frac{1}{z}\Phi$$

And by virtue of the property proved for this quantity we know that

$$o=\int\int\left(\rho\frac{d\Phi}{d\rho}+z\frac{d\Phi}{dz}+\frac{1}{z}\Phi\right)\rho\rho'\lambda\lambda'\,d\sigma d\sigma'$$

But we have

$$\mathrm{P}=\frac{1}{2\pi}\int\int\lambda'\rho'd\rho'dz'\Phi$$

therefore

$$\int\left(\rho\frac{d\mathrm{P}}{d\rho}+z\frac{d\mathrm{P}}{dz}+\frac{1}{z}\mathrm{P}\right)\rho\lambda d\sigma=o$$

Since now

$$\rho\frac{a\mathrm{P}}{d\rho}=\frac{d(\mathrm{P}\rho)}{d\rho}-\mathrm{P}$$

and

$$w\rho=\frac{d(\mathrm{P}\rho)}{d\rho},\quad s=-\frac{d\mathrm{P}}{dz}$$

this equation becomes

$$\int(w\rho-zs)\rho\lambda d\sigma-\frac{1}{z}\int\mathrm{P}\rho\lambda d\sigma=o$$

or

$$\int(w\rho-zs)\rho\lambda d\sigma=\frac{\mathrm{T}}{4\pi}.$$

We will now introduce the complementary modulus k' defined by

$$k'^2+k^2=1$$

we have for this modulus

$$k'^2=1-k^2=\frac{(z'-z)^2+(\rho'-\rho)^2}{(z'-z)^2+(\rho'+\rho)^2}$$

If k is very nearly equal to unity, that is, if k' is indefinitely small we see that Φ will be of the same order of magnitude as K, and again, that P will be of the same order as Φ. We will examine the order of K on the supposition that k' is infinitely small. Since k' is indefinitely small, we have, neglecting higher powers than the second,

$$k=\sqrt{1-k'^2}=1-\frac{1}{2}k'^2$$

Now

$$\mathrm{K}=\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

$$=\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-\sin^2\psi+k'^2\sin^2\psi}}$$

$$=\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{\cos^2\psi+k'^2\sin^2\psi}}$$

or, by introducing an indefinitely small quantity ε which is, nevertheless, indefinitely large as regards k',

$$\mathrm{K} = \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{\cos^2\psi + k'^2 \sin^2\psi}}$$

$$+\int_{o}^{\frac{\pi}{2}-\varepsilon} \frac{d\psi}{\sqrt{\cos^2\psi + k^2 \sin^2\psi}}$$

Write now in the first integral $\frac{\pi}{2} - \theta$ in in place of Ψ: since throughout the integral θ is small the integral becomes

$$\int_{o}^{\varepsilon} \frac{d\theta}{\sqrt{k'^2 + k^2\theta}}$$

$$= \frac{1}{k} \log \frac{k\varepsilon + \sqrt{k'^2 + k^2\varepsilon^2}}{k'}.$$

or since k' is infinitely small with regard to $k\varepsilon$ this is $= \frac{1}{k} \log \frac{2k\varepsilon}{k'}$ or $= \log \frac{2\varepsilon.}{k'}$

In the second integral $k' \sin\psi$ is throughout small as regards $\cos\psi$ and this integral is

$$\int_0^{\frac{\pi}{2}-\varepsilon} \frac{d\psi}{\cos\psi} = \log\tan\left\{\frac{1}{2}\pi - \frac{1}{2}\varepsilon\right\}$$

or what is the same thing$=\log\frac{2}{\varepsilon}$. Hence we have

$$\mathrm{K} = \log\frac{2\varepsilon}{k'} + \log\frac{2}{\varepsilon} = \log\frac{4}{k'}$$

Consequently for k' indefinitely small we see that K is indefinitely large, and Φ and therefore P are indefinitely large of the order $\log k'$. Of course Φ does not depend on E as for k nearly equal to unity we have,

$$\mathrm{E} = \int_0^{2\pi} \cos\psi\, d\psi = 1.$$

Representing the elliptic integral of the first kind as is usual by F, we have

$$\mathrm{F} = \int_0^{\psi} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

and also

$$\mathrm{E} = \int_0^{\psi} (1-k^2\sin^2\psi)^{\frac{1}{2}}\, d\psi.$$

Differentiating for k, and writing $\sqrt{1-k^2\sin^2\psi}=\Delta$

$$\frac{d\mathrm{F}}{dk}=\int\frac{k\sin^2\psi d\psi}{\Delta^3}$$

$$\frac{d\mathrm{E}}{dk}=-\int\frac{k\sin^2\psi d\psi}{\Delta}$$

Now writing $\sin^2\psi=\frac{1}{k^2}(1-\Delta^2)$ we see that these two integrals depend on

$$\int\frac{d\psi}{\Delta},\quad\int\Delta d\psi,\quad\int\frac{d\psi}{\Delta^3}:$$

the two first of these are F and E respectively; as regards the third, we have

$$\frac{d}{d\psi}\cdot\frac{\sin\psi\cos\psi}{\Delta}=\frac{1-2\sin^2\psi+k^2\sin^4\psi}{\Delta^3}$$

or

$$k^2\frac{d}{d\psi}\cdot\frac{\sin\psi\cos\psi}{\Delta}=\frac{\Delta^4-k'^2}{\Delta^3}=\Delta-\frac{k'^2}{\Delta^3}$$

and thence by integration,

$$\int\frac{d\psi}{\Delta^3}=\frac{1}{k'^2}\mathrm{E}-\frac{\mathrm{K}^2\sin\psi\cos\psi}{k'^2\Delta}$$

The expressions for $\frac{d\mathrm{E}}{dk}$, and $\frac{d\mathrm{F}}{dk}$, thus become,

$$\frac{dF}{dk}=\frac{1}{kk'^2}\left\{E-k'^2F\right\}-\frac{k\sin\psi\cos\psi}{dk'^2}$$

$$\frac{dE}{dk}=\frac{1}{k}\left\{E-F\right\}$$

and for the complete functions when $\psi=\frac{\pi}{2}$,

$$\frac{dE}{dk}=\frac{1}{kk'^2}[E-k'2F]$$

$$\frac{dE}{dk}=\frac{1}{k}[E-F]$$

When k' is indefinitely small the first of these is of the order $\frac{1}{k'^2}$ and the second by k'. Now we have obviously if k' be indefinitely small that $\frac{d\Phi}{dk}$ is of the order $\frac{1}{k'^2}$. We had also

$$\frac{dk}{dz}=\frac{k(z'-z)}{(z'-z)^2+(\rho'+\rho)^2}$$

$$\frac{dk}{d\rho}=\frac{k}{2\rho}\,\frac{(z'-z)^2+\rho'^2-\rho^2}{(z'-z)^2+(\rho'+\rho)^2}$$

These quantities are of the same order as k'. Therefore

$$\frac{d\Phi}{d\rho}-\frac{d\Phi}{dk}\,\frac{dk}{d\rho}\,\frac{1}{z}\,\frac{\Phi}{\rho}$$

and
$$\frac{d\Phi}{dz}=\frac{d\Phi}{dk}\frac{dk}{dz}$$

are of the same order as $\frac{1}{k'}$. By the aid of these preliminary investigations we will now proceed to the examination of the case when only one vortex ring exists in the fluid, and will further more suppose this ring to be of indefinitely small cross section and of the same order of magnitude as the indefinitely small quantity ε. We may again write as before

$$m=\int\lambda d\sigma$$

as m will be finite and as $d\sigma$ is of the order ε, λ must be of the order $\frac{1}{\varepsilon^2}$. Assume the equation of a circle such that the fluid elements of which it is composed shall lie indefinitely near the vortex filament. Let its equations be

$$\rho=\rho_0 \qquad z=z_0.$$

We had

$$P=\frac{1}{2\pi}\int\int\lambda'\rho' d'\rho dz'\Phi=\frac{m}{2\pi}\rho\rho_0\Phi$$
$$(z-z_0)\rho\rho_0)$$

for all points lying at a finite distance from the circle. For these points by aid of the equations

$$\Phi = \frac{z}{\sqrt{\rho\rho_0}}\left\{\left(\frac{2}{k} - k\right)\mathrm{K} - \frac{z}{k}\mathrm{E}\right\}$$

$$\rho s = -\frac{d(\mathrm{P}\rho)}{dz},\ w\rho = \frac{d(\mathrm{P}\rho)}{d\rho}$$

we can find the values of s and w. But a difficulty exists inasmuch as ρ_0 and z_0 are functions of the time and as such will have to be determined, and to that end, it is necessary to consider the points that lie indefinitely near the circle, or points in the vortex filament.

Suppose that the two circles $(\rho, z), (\rho_0, z_0)$ are at a distance apart, that is of the same order as ε. If we call r the distance between them, we have

$$r^2 = \rho^2 + \rho_0{}^2 - 2\rho\rho_0 \cos \vartheta + (z_0 - z)^2$$

or since ϑ is indefinitely small

$$r^2 = (\rho - \rho_0)^2 + (z - z_0)^2$$

this is of the same order as k'^2 therefore k' is of the same order as ε—therefore by our preceding investigations we see that P is of the same order as Φ when k'

is of the same order as ε. And also that P is of the order log. ε. We had for the energy

$$\mathrm{T}=2\pi\int \mathrm{P}\rho\lambda d\sigma$$

therefore T is of the same order as log. ε. The preceding investigations taken in connection with the equations

$$s\rho=-\frac{d(\mathrm{P}\rho)}{dz},\quad w\rho=\frac{d(\mathrm{P}\rho)}{d\rho}$$

show that inside the vortex ring s and w are of the same order as $\frac{1}{\varepsilon}$. We will now examine more closely ρ_0 and z_0. We have

$$\rho_0^2\int\lambda\sigma=\int\rho^2\lambda\,d\sigma$$
$$z_0\int\rho^2\lambda\,d\sigma=\int z\rho^2\lambda d\sigma$$

We assume that λ is constantly of the same sign, consequently the circle (ρ_0, z_0) lies either in or indefinitely near the vortex filament.

Now we found that, $\lambda\,d\,\sigma$ not varying with the time,

$$\int\rho^2\lambda d\sigma=\text{const.}$$

consequently ρ_0 does not vary with the time. That is, if only one vortex filament exists in the fluid its radius will remain unaltered, during the motion. Now to examine z_0. We have the equation

$$z_0\int \rho^2\lambda\, d\sigma = \int z\rho^2\lambda\, d\sigma$$

$$\int \lambda\, d\sigma = m$$

from these

$$m_0\rho^2 z_0 = \int z\rho^2\lambda\, d\sigma$$

Differentiating with respect to t

$$m\rho_0{}^2\frac{dz_0}{dt} = \int \rho^2\frac{dz}{dt}\lambda\, d\sigma + 2\int z\rho\frac{d\rho}{dt}\lambda\, d\sigma$$

But we had also

$$\frac{1}{4\pi}\mathrm{T} = \int (w\rho - sz)\rho\lambda\, d\sigma$$

$$= \int \rho^2\frac{dz}{dt}\lambda\, d\sigma - \int \rho^2\frac{d\rho}{dt}\lambda\, d\sigma$$

$$= m\rho_0{}^2\frac{dz_0}{dt} - 3\int z\rho\frac{d\rho}{dt}\lambda\, d\sigma$$

consequently

$$m\rho_0{}^2\frac{dz_0}{dt} = \frac{1}{4\pi}\mathrm{T} + 3\int z\rho\frac{d\rho}{dx}\lambda\, d\sigma$$

T, we have seen, is constant and infinitely great, of the same order as log ε; the difference between the values of z, in the second member, we know to be of the same order as ε, and we have seen that $\frac{d\rho}{d\tau}$ is of the order of $\frac{1}{\varepsilon}$; consequently the second member of the right hand side of the equation is finite. Therefore, it follows that $\frac{dz_0}{dt}$ is infinitely great of the order of log ε, and, neglecting the finite term — is constant. Since T is positive, $\frac{dz_0}{dt}$ must have the same sign as m, *i. e.*, as λ. Thus, if only one vortex ring exist in the fluid, it will retain its radius unaltered during the motion, and will advance in the direction of the axis of z with the velocity $\frac{dz_0}{dt}$. Now, as this vortex motion implies motion in all the particles of the fluid we have, that all the fluid particles at a finite distance from the filament flow through the ring in the direction of z, or the reverse, according to the sign λ. If $\vartheta = o$, we have

$\lambda = \eta$, and, according to the convention for positive rotation we have, that the motion in the direction of z will be positive, if, in the case $\theta = o$, η is positive. . Therefore, it follows that the ring is moving in the same direction as the fluid particles are flowing. I will now give the concluding remarks of Helmhotz's great memoir as nearly as may be. We can now readily see in general how two vortex rings having the same axis will move with reference to each other, by observing that each will have its motion modified, due by the motion of the particles of fluid, due to the rotation of the other. Suppose that both rings have the same direction of rotation, then they will both move forward in the same direction, and the former will widen and move more slowly, while the latter will contract and move forward more rapidly, finally overtaking and passing through the former, when the same operation will be repeated, the rings continually changing position throughout the motion.

Suppose that the vortex rings have

equal radii, the result is not changed in the case of the same direction of rotation existing for both. But now let them equal radii, and equal but opposite angular velocities; they will approach each other, and both will expand—approaching very near the effect of one upon the other is greatly increased—and they expand with constantly increasing velocity.

Now suppose that the rings having equal and opposite angular velocities, are symmetrical to each other. Then the motion in the direction of the axis of those particles that lie midway between the rings is *o*. We can conceive this surface in which these particles lie as a fixed boundary, and we have the case of vortex rings moving in contact with a fixed surface. These rings can be readily formed in water; or, rather, half rings can be formed, if we draw through the water rapidly, and for a short distance, a half immersed hemispherical vessel. Half rings will be formed in the water, having their axes in the fine surface of the fluid, which will move exactly as described in the theory. The free surface of the water will form a limiting plane, passing through the axes of the rings, and will not affect the motion. Rings of tobacco smoke have a rapid motion forwards in the direction of and due to the impulsive force which produced them; at the same time the ring flows through itself in the direction of the motion of translation.

It is very interesting to observe the motions of smoke rings, and for this purpose the following simple apparatus, which has been described in a great many places, will be found useful: A rough box, about ten inches long, and the same height and width, is large enough; one end of the box to be open, and over this stitch a piece of cloth or rubber; make a hole, about three inches in diameter, in the opposite end of the box, and a number of slides having smaller holes in them, to be placed over the larger opening and concentric with it. Now place inside of the box a vessel containing salt, on which pour strong sulphuric acid; and also place in the box a piece of cotton saturated with ammonia; fumes of ammonium chloride will immediately fill the box. Now tap on the stretched membrane; rings will issue from the hole in the slide at the opposite end, and will move forward with velocities proportional to the force of the blow struck. A very light tap is all that is necessary, and, indeed, is all that can be given, if it is desired to investigate the motion, as, otherwise, the rings move forward with such velocity that they can scarcely be followed with the eye. If the rings are allowed to impinge upon a surface, the rotational velocity is suddenly increased very much, and the rings thus spread out over the surface.

The same effects will be noticed if two rings be allowed to meet each other in their motion through the air. If the orifice be elliptic, the rings will be seen to interchange rapidly their axes, vibrating about a mean circular position.

If bubbles of phosphuretted hydrogen be allowed to escape into the air, each